Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

21.12.2017 Group Theory Exercise 9

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20p → 100 %

Hg. 1

(1)

$$SL(2,\mathbb{C}) = \{ A \in GL(2,\mathbb{C}) \mid \det A = 1 \}$$
$$GL(2,\mathbb{C}) = \{ A^{2\times 2} \mid A_{ij} \in \mathbb{C} \ \forall i,j,\ \det A \neq 0 \}$$

For complex matrices, we have

$\det(e^A) = e^{tr(A)}$ and

and $tr(X_i) = 0$ for our generators.

X doesn't have to be diagonizable?

$$\rightsquigarrow X_i = \begin{pmatrix} a_1 + i a_2 & b_1 + i b_2 \\ c_1 + i c_2 & -a_1 - i a_2 \end{pmatrix}$$

Allowing complex coefficients, we get

Are complex coefficients allowed?

$$X_1' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad X_2' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\quad X_3' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and thus by rearranging and building linear combinations

+2p

$$X_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(2)
$$[X_1, X_2] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2X_3$$
$$= i(-2i)X_3$$

$$[X_2, X_3] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2X_1$$
$$= i(-2i)X_1$$

$$[X_1, X_3] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2X_2$$
$$= i(-2i)X_2$$

$\rightsquigarrow$ $f_{123} = -2i = -f_{213}$, $f_{12b} = -f_{21b} = 0 \ \forall b \neq 3$

$f_{231} = -2i = -f_{321}$, $f_{23b} = -f_{32b} = 0 \ \forall b \neq 1$

$f_{132} = -2i = -f_{312}$, $f_{31b} = -f_{13b} = 0 \ \forall b \neq 2$

$f_{11b} = f_{22b} = f_{33b} = 0 \ \forall b$, as $[X_i, X_j] = i f_{ijk} X_k$

For the adjoint repr., we have $[ad(X_i)]_{jk} = [T_i]_{jk} = -i f_{ijk}$

$$\rightsquigarrow T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

bad if negative values here?

The killing form is defined by

$$B(X_i, X_j) = \mathrm{Tr}\left[ad(X_i)\, ad(X_j)\right]$$, we have

When exactly are structure constants antisym. in 2^{nd} and 3^{rd} index as well?

$$T_1 T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad T_1 T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T_1 T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix}, \quad T_2 T_2 = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$\mathrm{Tr}(T^a T^b) \neq \delta^{ab}$?

$$T_2 T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 0 \end{pmatrix}, \quad T_3 T_3 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow B(X_i, X_j) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

+4

Why can this eq always be fulfilled?

HÜ 2

$U\sigma_i U^\dagger = R_{ji}(U)\sigma_j$ definition for $R(U)$ where $U \in SU(2)$

$R(U)$: 3×3, U: 2×2

(1) $U\sigma_i U^\dagger = R_{ji}(U)\sigma_j \rightsquigarrow U\sigma_i^\dagger U^\dagger = R_{ji}^*(U)\sigma_j^\dagger$

$\overset{\sigma \text{ herm.}}{\Leftrightarrow} U\sigma_i U^\dagger = R_{ji}^*(U)\sigma_j$

Compare the coefficients or use $tr(\sigma^a\sigma^b) = \delta^{ab}$ again?

$\rightsquigarrow U\sigma_i U^\dagger = R_{ji}(U)\sigma_j = R_{ji}^*(U)\sigma_j$

$\rightsquigarrow R_{ji}^*(U) = R_{ji}(U) \rightsquigarrow R_{ji}(U) \in \mathbb{R} \;\forall j, i$

$\rightsquigarrow R(U)$ real

+

Now consider $R(UV)$:

$(UV)\sigma_i(UV)^\dagger = R_{ji}(UV)\sigma_j$

$\Leftrightarrow UV\sigma_i V^\dagger U^\dagger = R_{ji}(UV)\sigma_j$

$\Leftrightarrow R_{ji}(UV)\sigma_j = UV\sigma_i V^\dagger U^\dagger = U R_{ki}(V)\sigma_k U^\dagger$

$= R_{ki}(V)U\sigma_k U^\dagger = R_{ki}(V)R_{jk}(U)\sigma_j$

$= R_{jk}(U)R_{ki}(V)\sigma_j = (R(U)R(V))_{ji}\sigma_j$

$\rightsquigarrow R_{ji}(UV) = (R(U)R(V))_{ji}$

$\rightsquigarrow R(UV) = R(U)R(V)$

+3

(2) $SO(3) = \{M \in GL(3,\mathbb{R}) \mid M^T M = \mathbb{1},\ \det M = 1\}$

$U\sigma_i U^\dagger U\sigma_j U^\dagger = R_{ki}\sigma_k R_{\ell j}\sigma_\ell$

$\Leftrightarrow U\sigma_i\sigma_j U^\dagger = R^T_{ik}\sigma_k R_{\ell j}\sigma_\ell \quad , \sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k$

$\Leftrightarrow U(\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k)U^\dagger = R^T_{ik}R_{\ell j}(\delta_{k\ell}\mathbb{1} + i\epsilon_{k\ell m}\sigma_m) \quad (*)$

$\rightsquigarrow tr(U(\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k)U^\dagger) = R^T_{ik}R_{\ell j}\, tr(\delta_{k\ell}\mathbb{1} + i\epsilon_{k\ell m}\sigma_m)$

$\overset{tr(\sigma)=0}{\Leftrightarrow} 2\delta_{ij} = R^T_{ik}R_{\ell j}\cdot 2\delta_{k\ell}$

$\Leftrightarrow \delta_{ij} = (R^T R)_{ij}$

$\rightsquigarrow R^T R = \mathbb{1}$

Starting again from (*):

$U(\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k)U^\dagger = R^T_{ik} R_{\ell j}(\delta_{k\ell}\mathbb{1} + i\epsilon_{k\ell m}\sigma_m)$

and using the orthogonality just found

$$\rightsquigarrow\ i\epsilon_{ijk} U\sigma_k U^\dagger = i\epsilon_{k\ell m} R^T_{ik} R_{\ell j}\sigma_m$$

$$\Leftrightarrow\ \epsilon_{ijk} R_{nk}\sigma_n = \epsilon_{k\ell m} R_{ki} R_{\ell j}\sigma_m$$

$$\Leftrightarrow\ \epsilon_{ijm} R_{nm}\sigma_n = \epsilon_{k\ell n} R_{ki} R_{\ell j}\sigma_n$$

| taking tr(—) after multiplying w/ σ_a and rename $a \to n$ again

$$\Leftrightarrow\ \epsilon_{ijm} R_{nm} = \epsilon_{k\ell n} R_{ki} R_{\ell j}$$

$$\overset{\substack{\text{multiply} \\ \text{w/ transpose} \\ \text{+ sum over}}}{\Leftrightarrow}\ \epsilon_{ijm} R_{nm} R^T_{ia} R^T_{jb} = \epsilon_{k\ell n} R_{ki} R_{\ell j} R^T_{ia} R^T_{jb}$$

Allowed to multiply and then sum over all indices?

$$\Leftrightarrow\ \epsilon_{ijm} R_{nm} R_{ai} R_{bj} = \epsilon_{k\ell n}\delta_{ka}\delta_{\ell b}$$

$$\Leftrightarrow\ \epsilon_{ijm} R_{ai} R_{bj} R_{nm} = \epsilon_{abn}$$

$$\overset{\substack{\times \epsilon_{abn} \\ \text{+ sum over}}}{\Leftrightarrow}\ \epsilon_{abn}\epsilon_{ijm} R_{ai} R_{bj} R_{nm} = \epsilon_{abn}\epsilon_{abn} = 3!$$

$$\Leftrightarrow\ \underbrace{\frac{1}{3!}\epsilon_{abn}\epsilon_{ijm} R_{ai} R_{bj} R_{nm}}_{\det R} = 1$$

+ 4p

(3) We want to show that every $S \in SO(3)$ – where $S = S_3(\gamma) S_2(\beta) S_3(\alpha)$ can always be decomposed – has an inverse image in $SU(2)$ i.e. that $R: SU(2) \to SO(3)$ is surjective.

We will calculate the $R_{ji}(U)$ explicitly for

x real parameters always ?

$U = \exp(i x \sigma_j) \rightsquigarrow U^+ = \exp(-i x \sigma_j)$, fixed σ_j

$$U \sigma_i U^+ = (\sigma_i U + [U, \sigma_i]) U^+$$

$$= \sigma_i + [U, \sigma_i] U^+$$

$$[U, \sigma_i] = [e^{ix\sigma_j}, \sigma_i] = [(\cos x)\mathbb{1} + i(\sin x)\sigma_j, \sigma_i]$$

$$= i \sin(x) [\sigma_j, \sigma_i] = i \sin x \, (2i \epsilon_{jik} \sigma_k)$$

$$= + 2 \epsilon_{ijk} \sin(x) \sigma_k$$

$$= \sigma_i + 2\epsilon_{ijk} \sin x \, \sigma_k U^+$$

no sum over j to be taken?

$$= \sigma_i + 2\epsilon_{ijk} \sin x \, \sigma_k ((\cos x)\mathbb{1} - i(\sin x)\sigma_j)$$

$$= \sigma_i + 2\epsilon_{ijk} \sin x \cos x \, \sigma_k - 2i\epsilon_{ijk} \sin^2 x \, \sigma_k \sigma_j$$

$$= \sigma_i + \epsilon_{ijk} \sin(2x) \sigma_k + 2i \epsilon_{ijk} \Big\} \tfrac{1}{2}(1 - \cos(2x)) \Big\}\Big\{ \delta_{kj}\mathbb{1} + i\epsilon_{kjl}\sigma_l ($$

$$= \sigma_i + \epsilon_{ijk} \sin(2x) \sigma_k + \epsilon_{ijk} \epsilon_{kjl} (1 - \cos(2x)) \sigma_l$$

$$= \sigma_i + \epsilon_{ijk} \sin(2x) \sigma_k - \epsilon_{kji} \epsilon_{kjl} (1 - \cos(2x)) \sigma_l$$

$$= \sigma_i + \epsilon_{ijk} \sin(2x) \sigma_k - (\delta_{jj}\delta_{il} - \delta_{jl}\delta_{ij})(1 - \cos(2x))\sigma_l$$

$$= \sigma_i + \epsilon_{ijk} \sin(2x) \sigma_k - \sigma_i + \delta_{ij}\sigma_j + \cos(2x)\sigma_i - \cos(2x)\delta_{ij}\sigma_j$$

$$= \epsilon_{ijk} \sin(2x) \sigma_k + (1 - \cos(2x)) \delta_{ij} \sigma_j + \cos(2x) \sigma_i$$

$\rightsquigarrow$ $j=1$: $i=1$: $(1 - \cos(2x))\sigma_1 + \cos(2x)\sigma_1 = \sigma_1 \rightsquigarrow R_{11} = 1$

$R_{21} = 0$

$R_{31} = 0$

$i=2$: $\epsilon_{213} \sin(2x)\sigma_3 + \cos(2x)\sigma_2 \rightsquigarrow$ $R_{12} = 0$

$= \cos(2x)\sigma_2 - \sin(2x)\sigma_3$ $R_{22} = \cos(2x)$

$R_{32} = -\sin(2x)$

$\underline{i=3}$: $\epsilon_{312}\sin(2x)\sigma_2 + \cos(2x)\sigma_3 \rightsquigarrow R_{13} = 0$

$= \sin(2x)\sigma_2 + \cos(2x)\sigma_3$ $\quad R_{23} = \sin(2x)$

$R_{33} = \cos(2x)$

$$\rightsquigarrow \underset{\substack{\uparrow \\ \text{generator}}}{R(U_{\sigma_1})} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2x) & \sin(2x) \\ 0 & -\sin(2x) & \cos(2x) \end{pmatrix} \hat{=} \text{ rotation around x-axis w} \left(x = -\frac{\omega}{2}\right)$$

$\underline{j=2}$, $\underline{i=1}$: $\epsilon_{123}\sin(2x)\sigma_3 + \cos(2x)\sigma_1 \rightsquigarrow R_{11} = \cos(2x)$

$= \sin(2x)\sigma_3 + \cos(2x)\sigma_1$ $\quad R_{21} = 0$, $R_{31} = \sin(2x)$

Rotation around x-axis not needed?

$\underline{i=2}$: $(1-\cos(2x))\sigma_2 + \cos(2x)\sigma_2 = \sigma_2 \rightsquigarrow R_{12} = 0$

$R_{22} = 1$

$R_{32} = 0$

$\underline{i=3}$: $\epsilon_{321}\sin(2x)\sigma_1 + \cos(2x)\sigma_3 \rightsquigarrow R_{13} = -\sin(2x)$

$= -\sin(2x)\sigma_1 + \cos(2x)\sigma_3$ $\quad R_{23} = 0$

$R_{33} = \cos(2x)$

$$\rightsquigarrow R(U_{\sigma_2}) = \begin{pmatrix} \cos(2x) & 0 & -\sin(2x) \\ 0 & 1 & 0 \\ \sin(2x) & 0 & \cos(2x) \end{pmatrix} \quad x = -\frac{\beta}{2} \text{ rot. around y-axis}$$

$\underline{j=3}$, $\underline{i=1}$: $\epsilon_{132}\sin(2x)\sigma_2 + \cos(2x)\sigma_1 \rightsquigarrow R_{11} = \cos(2x)$

$= -\sin(2x)\sigma_2 + \cos(2x)\sigma_1$ $\quad R_{21} = -\sin(2x)$

$R_{31} = 0$

$\underline{i=2}$: $\epsilon_{231}\sin(2x)\sigma_1 + \cos(2x)\sigma_2 \rightsquigarrow R_{12} = \sin(2x)$

$= \sin(2x)\sigma_1 + \cos(2x)\sigma_2$ $\quad R_{22} = \cos(2x)$

$R_{32} = 0$

$\underline{i=3}$: $(1-\cos(2x))\sigma_3 + \cos(2x)\sigma_3 = \sigma_3 \rightsquigarrow R_{13} = 0$

$R_{23} = 0$

$R_{33} = 1$

$$\rightsquigarrow R(U_{\sigma_3}) = \begin{pmatrix} \cos(2x) & \sin(2x) & 0 \\ -\sin(2x) & \cos(2x) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad x = -\frac{\alpha}{2}, -\frac{\gamma}{2} \text{ rot. around z-axis}$$

Using (1): $R(UV) = R(U)R(V)$, it immediately follows

that $S = S_3(\gamma)S_2(\beta)S_3(\alpha) = R(U_{\sigma_3}^{\gamma}).R(U_{\sigma_2}^{\beta}).R(U_{\sigma_3}^{\alpha})$

$$= R(\underbrace{U_{\sigma_3}^{\gamma}\, U_{\sigma_2}^{\beta}\, U_{\sigma_3}^{\alpha}}_{\in SU(2)}) = R(U_{rot}), \text{ where } U_{rot} \in SU(2)$$

+4p

(4) Let $R(U_1) = R(U_2) \rightsquigarrow U_1\sigma_i U_1^+ = U_2\sigma_i U_2^+ \quad \forall i$

$\Leftrightarrow \sigma_i U_1^+ U_2 = U_1^+ U_2 \sigma_i \rightsquigarrow [\sigma_i, U_1^+U_2] = 0$

$\xrightarrow{\text{Schur's 2nd Lemma}} U_1^+ U_2 = \lambda \mathbb{1} \rightsquigarrow \det(U_1^+U_2) = \lambda^2 = \det(U_1^+)\det(U_2) = 1$

$\rightsquigarrow \lambda = \pm 1$ and therefore $U_1^+ U_2 = \pm \mathbb{1} \rightsquigarrow U_1^+ = \pm U_2^{-1} = U_1^+$

$\rightsquigarrow U_1 = \pm U_2$

+3