

## Disclaimer

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<https://www.physics-and-stuff.com/>

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12.10.2017 Group theory 1st Exercise

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A.1.1. We will start with (2):

let  $g'g \stackrel{(*)1}{=} e$  and  $g''g' \stackrel{(*)2}{=} e$  be the left inverses

$$\rightarrow gg' \stackrel{(*)2}{=} (g''g')(gg') = (g''(gg')) \stackrel{(*)1}{=} g''g' = e \quad \square$$

and thus  $g'$  is also the right inverse to  $g$ .  $(*)1'$

Now assume:  $\tilde{g}'g \stackrel{(*)3}{=} e = gg' \stackrel{(*)1}{=} e$  furthermore; then

$$\rightarrow \tilde{g}' \stackrel{(*)1}{=} \tilde{g}'(gg') \stackrel{(*)3}{=} eg' = g' \quad \square$$

(1) is now easy, as with  $eg \stackrel{(*)4}{=} g \forall g \in G$  for some  $e \in G$ , we have

$$ge \stackrel{(*)1}{=} gg'g \stackrel{(*)1}{=} eg = g \quad \square \quad (**)4'$$

and if we have  $\tilde{e}g \stackrel{(*)5}{=} g \stackrel{(*)5}{=} g\tilde{e} \forall g \in G$  for some  $\tilde{e}$ ,

$$\text{we get: } e\tilde{e} \stackrel{(*)4}{=} \tilde{e} \stackrel{(*)4'}{=} \tilde{e}e \stackrel{(*)5}{=} e \quad \square$$

Point groups  
= Sym. groups?

no point groups  
are groups of  
geometric sym.  
that keep at least  
one point fixed  
like sym group  
e.g.  $C_n$

A.1.2

i) 

	e	c
e	e	c
c	c	e

$$\rightarrow T_{ij} = \begin{pmatrix} e & c \\ e & e \end{pmatrix}$$

ii)

	e	c	c <sup>2</sup>
e	e	c	c <sup>2</sup>
c	c	c <sup>2</sup>	e
c <sup>2</sup>	c <sup>2</sup>	e	c

$$\rightarrow T_{ij} = \begin{pmatrix} e & c & c^2 \\ c & c^2 & e \\ e^2 & e & c \end{pmatrix}$$

iii)

	e	c	b	bc
e	e	c	b	bc
c	c	e	bc	b
b	b	bc	e	c
bc	bc	b	c	e

•  $(bc) = (bc)^{-1} = c^{-1}b^{-1}$   
 $\Leftrightarrow b^{-1} = ebc$  but we know  
 $b^{-1} = b$ , thus:  $b = ebc$   
 and using  $c^2 = e$ :  $cb = bc$

$$\rightarrow T_{ij} = \begin{pmatrix} e & c & b & bc \\ c & e & bc & b \\ b & bc & e & c \\ bc & b & c & e \end{pmatrix}$$



iv)

	e	c	c <sup>2</sup>	b	bc	bc <sup>2</sup>
e	e	c	c <sup>2</sup>	b	bc	bc <sup>2</sup>
c	c	c <sup>2</sup>	e	bc <sup>2</sup>	b	bc
c <sup>2</sup>	c <sup>2</sup>	e	c	bc	bc <sup>2</sup>	b
b	b	bc	bc <sup>2</sup>	e	c	c <sup>2</sup>
bc	bc	bc <sup>2</sup>	b	c <sup>2</sup>	e	c
bc <sup>2</sup>	bc <sup>2</sup>	b	bc	c	c <sup>2</sup>	e

D<sub>3</sub>, D<sub>2</sub> ✓  
 always look like this?  
 Or possible that c<sup>2</sup> = e? e.g.  
 D<sub>3</sub>, D<sub>2</sub> etc. actually look like this by definition & others called differently

•  $(bc) = (bc)^{-1} = c^{-1}b^{-1} \Leftrightarrow cbc = b^{-1}$ , but we know  $b^{-1} = b$   
 $\rightarrow cbc = b$

• using  $c^3 = e$  and the above, we find  $bc = c^2b$   
 and also  $cb = bc^2$

$\rightarrow T_{ij} = \begin{pmatrix} e & c & c^2 & b & bc & bc^2 \\ c & c^2 & e & bc^2 & b & bc \\ c^2 & e & c & bc & bc^2 & b \\ b & bc & bc^2 & e & c & c^2 \\ bc & bc^2 & b & c^2 & e & c \\ bc^2 & b & bc & c & c^2 & e \end{pmatrix}$

The groups C<sub>2</sub>, C<sub>3</sub> and D<sub>2</sub> are abelian. The corresponding matrices are symmetric, while the group table for D<sub>3</sub> isn't symmetric and D<sub>3</sub> isn't abelian. It's obvious that

T<sub>ij</sub> symmetric  $\Leftrightarrow$  corresponding group abelian holds always! ✓

A.1.3: let's look at a group of order n:  $\{g_1, \dots, g_n\} = G$

If we ~~then~~ now look at the product of two elements, the closure axiom tells us that  $g_i g_j \in G$  and thus  $g_i g_j = g_k$ .

Let's now fix some  $g_\ell$  and assume that for  $m \neq n$

$g_k = g_\ell g_m = g_\ell g_n \rightarrow g_m = g_n$  by multiplying with  $g_\ell^{-1}$  from the left. Thus in one row, each element  $g_k$  can only appear

at most one time. But we also have n operations/elements in the row to multiply with  $g_\ell \rightarrow$  each element appears exactly once.

Analogue for  $g_m g_\ell = g_n g_\ell$  for the columns! ✓

Extra argument for columns needed?