Disclaimer

12.10.2017 **Group theory 1st Exercise** Marvin Zanke

A 1.1. We will start with (2):

let $g'g \underset{(*1)}{=} e$ and $g''g' \underset{(*2)}{=} e$ be the left inverses

$$\rightsquigarrow\ gg' \underset{(*2)}{=} (g''g')(gg') = (g''(g'g)g') \underset{(*1)}{=} g''g' = e \quad \blacksquare$$

and thus g' is also the right inverse to g. $(*1')$

Now assume: $\tilde{g}'g \underset{(*3)}{=} e \underset{(*3)}{=} g\tilde{g}'$ furthermore; then

$$\rightsquigarrow\ \tilde{g}' \underset{(*1')}{=} \tilde{g}'(gg') \underset{(*3)}{=} eg' = g' \quad \blacksquare$$

(1) is now easy, as with $eg \underset{(*4)}{=} g \ \forall g \in G$ for some $e \in G$, we have

$$ge \underset{(*1)}{=} gg'g \underset{(*1')}{=} eg = g \quad \blacksquare \qquad (*4')$$

and if we have $\tilde{e}g \overset{(*5)}{=} g \overset{?}{=} g\tilde{e} \ \forall g \in G$ for some $\tilde{e}$,

we get: $e\tilde{e} \underset{(*4)}{=} \tilde{e} \underset{(*4')}{=} \tilde{e}e \underset{(*5)}{=} e$ $\blacksquare$

Point groups $\subseteq$ sym. groups?
$\rightsquigarrow$ point groups are groups of geometric sym. that keep at least one point fixed
while sym group e.g. C_n

A 1.2

i)

$\cdot$	e	c
e	e	c
c	c	e

$$\rightsquigarrow\ T_{ij} = \begin{pmatrix} e & c \\ c & e \end{pmatrix}$$

ii)

	e	c	c^2
e	e	c	c^2
c	c	c^2	e
c^2	c^2	e	c

$$\rightsquigarrow\ T_{ij} = \begin{pmatrix} e & c & c^2 \\ c & c^2 & e \\ c^2 & e & c \end{pmatrix}$$

iii)

	e	c	b	bc
e	e	c	b	bc
c	c	e	bc	b
b	b	bc	e	c
bc	bc	b	c	e

- $(bc) = (bc)^{-1} = c^{-1}b^{-1}$

$\Leftrightarrow b^{-1} = cbc$ but we know $b^{-1} = b$, thus: $\underline{b = cbc}$

and using $c^2 = e$: $\underline{cb = bc}$

$$\rightsquigarrow\ T_{ij} = \begin{pmatrix} e & c & b & bc \\ c & e & bc & b \\ b & bc & e & c \\ bc & b & c & e \end{pmatrix}$$

iv)

	e	c	c^2	b	bc	bc^2
e	e	c	c^2	b	bc	bc^2
c	c	c^2	e	bc^2	b	bc
c^2	c^2	e	c	bc	bc^2	b
b	b	bc	bc^2	e	c	c^2
bc	bc	bc^2	b	c^2	e	c
bc^2	bc^2	b	bc	c	c^2	e

D_3, D_2 ✓ always look like this? Or possible that e.g. $c^2 = e$?

→ D_3, D_2 etc. actually look like this by definition & others called differently

• $(bc) = (bc)^{-1} = c^{-1}b^{-1} \Leftrightarrow cbc = b^{-1}$, but we know $b^{-1} = b$

$\rightsquigarrow cbc = b$.

• using $c^3 = e$ and the above, we find $bc = c^2 b$

and also $cb = bc^2$

$$\rightsquigarrow T_{ij} = \begin{pmatrix} e & c & c^2 & b & bc & bc^2 \\ c & c^2 & e & bc^2 & b & bc \\ c^2 & e & c & bc & bc^2 & b \\ b & bc & bc^2 & e & c & c^2 \\ bc & bc^2 & b & c^2 & e & c \\ bc & b & bc & c & c^2 & e \end{pmatrix}$$

The groups C_2, C_3 and D_2 are abelian. The corresponding matrices are symmetric, while the group table for D_3 isn't symmetric and D_3 isn't abelian. It's obvious that T_{ij} symmetric $\Leftrightarrow$ corresponding group abelian holds always ∎

A 1.3: let's look at a group of order n: $\{g_1, \dots, g_n\} = G$

If we ~~...~~ now look at the product of two elements, the closure axiom tells us that $g_i g_j \in G$ and thus $g_i g_j = g_k$.

Let's now fix some g_ℓ and assume that for $m \neq n$

$g_k = g_\ell g_m = g_\ell g_n \rightsquigarrow g_m = g_n$ by multiplying with g_ℓ^{-1} from the left. Thus in one row, each element g_k can only appear at most one time. But we also have n operations / elements in the row to multiply with g_ℓ $\rightsquigarrow$ each element appears exactly once.

Analogue for $g_m g_\ell = g_n g_\ell$ for the columns ∎

Extra argument for columns needed?