Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check: https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

This work by Marvin Zanke is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.
21. lesson  
Group theory  
2nd Exercise

Ad 1.

(1) Consider $D_{4h}$. Let's denote $O = A_{h}$

We claim that: $D_2 = C^3 D_4^n$, $D_3 = C^2 D_4^n$, $D_4 = C D_4^n$.

and show these relations on $O$.

$D_2 O = \begin{array}{c} A \hline B \\ C \hline D \end{array}$, while $C^3 D_4 A = \begin{array}{c} A \hline B \\ C \hline D \end{array}$

$D_3 O = \begin{array}{c} D \hline C \\ B \hline A \end{array}$, while $C^2 D_4 A = \begin{array}{c} D \hline C \\ B \hline A \end{array}$

$D_4 O = \begin{array}{c} B \hline D \\ A \hline C \end{array}$, while $C D_4 A = \begin{array}{c} B \hline D \\ A \hline C \end{array}$

(2) $D_n = \langle b, c \mid b^2 = c^n = (bc)^2 = e \rangle$

The order of $D_n$ can easily be counted as:

$b, b^2 = e, c, c^2, \ldots, c^{n-1}, bc, bc^2, \ldots, bc^{n-1}$

\[ N+1 + N-1 = 2N = |D_n| \]

As $(bc)^2 = e$, we find: $b c = (bc)^{-1} = c^{-1} b^{-1} = c^{n-1} b^{-1}$

$\Rightarrow c \cdot b = b \Rightarrow c b = b c^{n-1}$

Let's denote \( \chi = \begin{array}{c} 1 \\ N-1 \\ 2 \end{array} \) and distinguish

between the 2 cases, where the reflected axis cuts through a side and where the axis cuts between 2 sides (see $X$ and $\phi$, respectively). We now show the reason to demand these conditions.
\[
\begin{align*}
\delta c \chi &= \delta N \quad = \quad N \quad = \quad N \quad = \quad N \quad = \quad N \\
\delta c \phi &= \delta N \quad = \quad N \quad = \quad N \quad = \quad N \\
\alpha b \chi &= c N \quad = \quad N \quad = \quad N \quad = \quad N \\
\alpha b \phi &= c N \quad = \quad N \quad = \quad N \quad = \quad N
\end{align*}
\]
Order of \( S_n \) is \( n! \)

2 generators

\[ S_3 = \langle (12), (23) \rangle \]

\[ (1623)(3645)(1342) = (1 2 3 4 5 6) \]

Other way: start with \( (16) \rightarrow 3 \rightarrow 6 \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow (126) \)

\[ (1 3 \ 4 13) (54321) (32)(135) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 4 & 5 \end{pmatrix} \]

\[ (13 \ 413 \ 34)(4543)(32)(32)(135) = (34)(43)(34)(135) \]

\[ (345)(135) = (34)(31 \ 43)(13)(43) = e \]

\( \{123456\} \) cycles common, as no elements in common

\[ [123456]^{30} \leq [123456]^{30} \]

\[ = e \cdot e = e, \quad \text{as} \quad [123456]^{6} = e \quad \text{and} \quad [78910]^{5} = e \]

\[ [78910]^{32} = \frac{(135)(246)(78910)}{23456 \ 12} \]