Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

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02.11.2017 Group theory Exercise 3 Martin Zemlka

A 3.1

Group G, $|G| = N_G$, $g \in G$, $g \neq e$

(1) $g^m = e$ with $m = \min\{k \mid g^k = e\}$

Construct a subgroup $H = \{g^i \mid i = 1, \ldots, m\}$

$\rightsquigarrow |H| = m$ and from Lagrange theorem

$|G| \bmod |H| = 0$ and thus

$N_G \bmod m = 0 \rightsquigarrow \frac{N_G}{m} = c$

(2) Let's consider a Group G with $|G| = p$ prime

We can choose $e \neq g \in G$ and construct a subgroup

$H = \langle g \mid g^N = e \rangle$. We know $|H| \geq 2$, but

$|H|$ needs to be a divisor of $|G|$ and thus $|H| = p$.

Thus G is generated by g and is cyclic.

Example: $\mathbb{Z}_p$

(3) Groups with at most 4 elements are abelian

$\rightsquigarrow \mathbb{Z}_3$ as 3 prime has no proper subgroup

(4) $H \subset G$ normal subgroup: $gH = Hg \;\forall g \in G$

$gh_1g^{-1} \in H$ for $h \in H$

$\rightsquigarrow gh_1g^{-1} = h_1'$

$gh_2g^{-1} = h_1' \rightsquigarrow g^{-1}h_1'g = h_2 = g^{-1}gh_1g^{-1}g = h_1$

$\rightsquigarrow gHg^{-1} = H$ as injectivity implies surjectivity for equal cardinality.

(5) $(g_1 H)(g_2 H) := g_1 g_2 H \quad \forall g_1, g_2 \in G$ will $H \subset G$

Assume $\exists g'$ s.t. $g'H \neq Hg' \rightsquigarrow (g')^{-1}H \neq H(g')^{-1}$

$\rightsquigarrow$

A3.2

$$\{(b_1,0),\ldots,(b_n,0)\}$$

$$(H_1 \times H_2)/H_1 := \{(x_i,y_j) + H_1 \mid (x_i,y_j) \in (H_1 \times H_2)\}$$

H_1: n elements, H_2: m elements

Map $\{a_i\} \mapsto \{\{(x_1+b_1, a_i), \ldots (x_1+b_n, a_i)\}, \ldots,$
$i=1,\ldots,m$, $a_i \in H_2$ $\{(x_n+b_1, a_i), \ldots, (x_n+b_n, a_i)\}\}$

Isomorphism $\simeq H_2$

A.3.3

(1) $Z(G)$ is abelian by construction and inherits associativity from G.

- $e \in Z(G)$ as $eg = ge$ $\forall g \in G$
- $a \in Z(G) \Rightarrow ag = ga$ $\forall g \in G \Leftrightarrow g^{-1}a = ag^{-1}$ $\forall g \in G$ and thus $g^{-1} \in G$
- $a,b \in Z(G) \rightsquigarrow ag = ga$, $bg = gb$ $\forall g \in G$
  $\Rightarrow (ab)g = agb = g(ab)$ $\forall g \in G$

(2) $gZ(G)g^{-1} \overset{!}{=} Z(G)$ $\forall g \in G$
$(gzg^{-1}h) = zh = hz = hgzg^{-1}$ & other way trivial

(3) $Z(G) < G$ proper. Assume G abelian $\Rightarrow ab = ba$ $\forall a,b \in G$
$\rightsquigarrow Z(G) = G \rightarrow Z(G)$ not a proper subgroup

4)

A3.5 $|H| = |G|/2 \Rightarrow \frac{|G|}{|H|} = 2$

$\Rightarrow G = \bigcup_{g \in G} gH = \bigcup_{g \in G} Hg$

$= eH \cup gH = He \cup Hg$ for $g \notin H$

$\Rightarrow gH = Hg$ as $hH = Hh$ for $h \in H$ trivial