Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

15.11.2011 **Group Theory Exercise 4** Marvin Zanke

A4.1

$(a_1, \Lambda_1) \cdot (a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$

for $P = \{(a, \Lambda)\}$, $\Lambda \in \mathcal{L}$, $a \in \mathbb{R}^4$

(1) • closure: $(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \in P$,

as $a_1 + \Lambda_1 a_2 \in \mathbb{R}^4$ and $\Lambda_1 \Lambda_2 \in \mathcal{L}$ because it's a group

• associativity: $(a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)(a_3, \Lambda_3) = (a_1 + \Lambda_1 a_2 + \Lambda_1 \Lambda_2 a_3, \Lambda_1 \Lambda_2 \Lambda_3)$

$(a_1, \Lambda_1)(a_2 + \Lambda_2 a_3, \Lambda_2 \Lambda_3) = (a_1 + \Lambda_1 (a_2 + \Lambda_2 a_3), \Lambda_1 \Lambda_2 \Lambda_3)$

$= (a_1 + \Lambda_1 a_2 + \Lambda_1 \Lambda_2 a_3, \Lambda_1 \Lambda_2 \Lambda_3)$

• neutral element: $(0, \mathbb{1})$ as $\mathbb{1} \in \mathcal{L}$

and $(0, \mathbb{1})(a, \Lambda) = (a, \Lambda)$

• inverse: $(a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) \overset{!}{=} (0, \mathbb{1})$ with $(a_2, \Lambda_2) = (a_1, \Lambda_1)^{-1}$

$\rightsquigarrow \Lambda_2 = \Lambda_1^{-1}$, $a_2 = -\Lambda_1^{-1} a_1$

$\rightsquigarrow (a_1, \Lambda_1)^{-1} = (-\Lambda_1^{-1} a_1, \Lambda_1^{-1}) \in P$ as $-\Lambda_1^{-1} a_1 \in \mathbb{R}^4$

and $\Lambda_1^{-1} \in \mathcal{L}$

(2) $T_4 = \{(a, \mathbb{1})\}$ is a subgroup, as

Minkowski space $\mathbb{R}^4$?

• closure: $(a_1, \mathbb{1}) \cdot (a_2, \mathbb{1}) = (a_1 + a_2, \mathbb{1}) \in T_4$.

• associativity is inherited from P

• $(0, \mathbb{1}) \in T_4$ obviously

• inverse: $(a, \mathbb{1})(-a, \mathbb{1}) = (a - a, \mathbb{1}) = (0, \mathbb{1})$

and $(-a, \mathbb{1}) \in T_4$ obvious.

(3) We need to show that $g h g^{-1} \in T_4 \ \forall g \in P, h \in T_4$

$(a_1, \Lambda_1)(b, \mathbb{1})(-\Lambda_1^{-1} a_1, \Lambda_1^{-1})$

$= (a_1, \Lambda_1)(b - \Lambda_1^{-1} a_1, \Lambda_1^{-1}) = (a_1 + \Lambda_1 (b - \Lambda_1^{-1} a_1), \mathbb{1})$

$= (\Lambda_1 b, \mathbb{1}) \in T_4$.

T_4 is not countable?

(4) $P/T_4 \xrightarrow{f} \mathcal{L}$, where $P/T_4 \ni (a_1, \Lambda_1) T_4 \xrightarrow{f} \Lambda_1 \in \mathcal{L}$

is obviously surjective, as every $\Lambda \in \mathcal{L}$ is in $\{(a, \Lambda) T_4 \mid (a, \Lambda) \in P\}$

To show injectivity, we will consider $\ker f = \{ g \in P/T_4 \mid f(g) = \mathbb{1} \}$

the neutral element in P/T_4 is $(0,\mathbb{1})T_4 = (0,\mathbb{1})(a,\mathbb{1})T_4$
$$= (a,\mathbb{1})T_4$$

because $(0,\mathbb{1})T_4 \circ (a,\Lambda) = (a,\Lambda)(0,\mathbb{1})T_4 = (a,\Lambda)T_4$

$f((0,\mathbb{1})T_4) = \mathbb{1}$ by construction and $f((a,\Lambda)T_4) = f((0,\mathbb{1})T_4) = \mathbb{1}$

but $f((a,\Lambda)T_4) = \Lambda \neq \mathbb{1}$ and thus

$\ker f = \{ (0,\mathbb{1})T_4 \} = \{ (a,\mathbb{1})T_4 \}$

Furthermore, we have to show that the mapping is a homomorphism:

$$f((a_1,\Lambda_1)(a_2,\Lambda_2)T_4) = f((a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2)T_4) = \Lambda_1\Lambda_2$$
$$= f((a_1,\Lambda_1)T_4)\, f((a_2,\Lambda_2)T_4)$$

(5) P is not semisimple (no abelian inv. SG), as T_4 is an invariant SG, which obviously is abelian.

Other inv. SG of P?

A4.2

(1) $|S_4| = 24 \rightsquigarrow$ SG H, $|H| \in \{1, 2, 3, 4, 6, 8, 12, 24\}$

	Y.F.	Cycle str.	P	#	ord	regular
[4]		$(*)(*)(*)(*)$	+	1	1	✗
[3,1]		$(**)(*)(*)$	−	6	2	✗
[2,2]		$(**)(**)$	+	3	2	✓
[2,1,1]		$(***)(*)$	+	8	3	✗
$[1^4]$		$(****)$	−	6	4	✓

(2) Invariant SG need to be the union of conjugacy classes ($\hat{=}$ cycle str.)

$\rightsquigarrow H_1 = \{e\} = \{(1)(2)(3)(4)\}$ is an inv. (non-proper) SG

$H_3 = \{(12)(34), (13)(24), (14)(23), e\}$ $\quad (12)(34)(31)(24) = (23)(14)$

$H_{12} = \{(***)(*), (**)(**), e\}$ where the $*$'s represent all possible elements in the conjugacy class $\}$

$H_3 \shortparallel$, $H_{12} \shortparallel A_4$

$S_4/H_{12} = S_4/A_4 \cong C_2$

$S_4/H_3 \cong S_4/Z_2 \times Z_2 \cong S_3$ $\quad \Big|\ ?$

Non invariant e.g. $K = \{e, (12)(34), (12), (34)\}$

(3) order 4: isomorphic to Z_4 $(\cong C_4)$ or $Z_2 \times Z_2$

$C_4 = \{e, (1234), (13)(24), (1432)\}$ + 2 more

$Z_2 \times Z_2 = \{e, (12)(34), (13)(24), (23)(41)\}$

$Z_2 \times Z_2 = \{e, (12), (34), (12)(34)\}$ + 2 more

A4.3

(1) $D_4 = \langle c, b \mid c^4 = e, b^2 = e, (bc)^2 = e \rangle$

$$Z(D_4) = \{ z \in D_4 \mid zg = gz \ \forall g \in D_4 \}$$

$$z \in Z(D_4) \Leftrightarrow z = gzg^{-1} \ \forall g \in D_4$$

Pick $b \in D_4 \rightsquigarrow b\, b^k c^l b \stackrel{!}{=} b^k c^l$

$\Leftrightarrow b^k c^{-l} \stackrel{!}{=} b^k c^l \Leftrightarrow c^{2l} = e \Leftrightarrow l = 0, 2$

Now $\cdot\ c^i c^{2l} c^{-i} = c^{2l}$ trivial

$\cdot\ c^i b c^{2l} c^{-i} \stackrel{!}{=} bc^{2l} \Leftrightarrow c^{2i} bc^{2l} = bc^{2l} \Leftrightarrow c^{2i} = e$

$\Leftrightarrow i = 0, 2$

but should hold $\forall i$

$\rightsquigarrow Z(D_4) = \{e, c^2\}$

Where does it fail to construct this isomorphism?

- D_4 is non abelian
- $|D_4 / Z(D_4)| = 4$ is abelian as every group with at most 4 elements is
- $|Z(D_4)| = 2$ is abelian

$\rightsquigarrow D_4 / Z(D_4) \times Z(D_4)$ is abelian, but D_4 isn't $\lightning$

$\rightsquigarrow$ no isomorphism $\lightning$

(2) $G \xrightarrow{t_{g_0}} G$, $g \overset{t_{g_0}}{\mapsto} g_0 g g_0^{-1}$, $g_0 \in G$

- homomorphism: $t_{g_0}(g_1 g_2) = g_0 g_1 g_2 g_0^{-1} = g_0 g_1 g_0^{-1} g_0 g_2 g_0^{-1}$

$= t_{g_0}(g_1) \cdot t_{g_0}(g_2)$

Rearrangement theorem working?

- Surjective: $\forall g' \in G \ \exists g$ s.t. $t_{g_0}(g) = g'$; choose $g = g_0^{-1} g' g_0$

$\rightsquigarrow t_{g_0}(g) = g_0 g_0^{-1} g' g_0 g_0^{-1} = g'$

- injective: $t_{g_0}(g_1) = t_{g_0}(g_2) \Leftrightarrow g_0 g_1 g_0^{-1} = g_0 g_2 g_0^{-1}$

$\Leftrightarrow g_1 = g_2$

$I(G) = \{t_{g_0} \mid g_0 \in G\}$

- Closure: $t_{g_1} \circ t_{g_2}(g) = t_{g_1}(g_2 g g_2^{-1}) = g_1 g_2 g \underbrace{g_2^{-1} g_1^{-1}}_{(g_1 g_2)^{-1}}$
$= t_{g_1 g_2}(g) \in I(G)$ as $g_1 g_2 \in G$
- associativity: $(t_{g_1} \circ t_{g_2}) t_{g_3}(g) = t_{g_1 g_2} t_{g_3}(g) = t_{(g_1 g_2) g_3}(g)$
$= t_{g_1(g_2 g_3)}(g) = t_{g_1} \circ (t_{g_2} \circ t_{g_3})(g)$ as G associative
- neutral element: $t_e \in I(G)$ as $e \in G$ and
$t_e \circ t_{g_0}(g) = t_{e \cdot g_0}(g) = t_{g_0}(g)$
- inverse: $t_{g_0^{-1}}(g) \in I(G)$ as $g_0 \in G \Rightarrow g_0^{-1} \in G$
as $t_{g_0^{-1}} \circ t_{g_0}(g) = t_{g_0^{-1} g_0}(g) = t_e(g)$

$G/Z(G) = \{g Z(G) \mid g \in G\}$

$I(G) \xrightarrow{\varphi} G/Z(G)$, $t_g \xmapsto{\varphi} g Z(G)$

obviously surjective by construction

$\varphi(t_{g_1} \circ t_{g_2}) = \varphi(t_{g_1 g_2}) = g_1 g_2 Z(G) = g_1 Z(G)\, g_2 Z(G) = \varphi(t_{g_1}) \varphi(t_{g_2})$
as $Z(G)$ (always) invariant

φ is also 1:1, as $\ker\varphi = \{i \in I(G) \mid \varphi(i) = e = eZ(G)\}$

$\varphi(t_e) = eZ(G) \Rightarrow t_e \in \ker\varphi$

$\varphi(t_g) = gZ(G) = eZ(G)$ iff $g \in Z(G)$ but then,

$t_g(k) = gkg^{-1} = k = eke^{-1} = t_e(k)$

Iff? Not possible otherwise?