Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

21.11.2017 Group Theory Exercise 5

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A.5.1

(1) $G/N = \{gN \mid g \in G\}$

$D^{G/N}$ a representation of this quotient group, i.e. $D^{G/N}(gN)$ on L.

Define $D^G(g) := D^{G/N}(gN)$.

This obviously acts on L as well and it forms a group if $D^{G/N}(gN)$ forms a group.

Group prop. have to be shown? Why does it even have to be a group? ⇝ $D^{G/N}$ already a representation and thus group?! Why need faithful?

Furthermore, $D^G(g_1 \circ g_2) = D^{G/N}(g_1 g_2 N) \overset{\text{Def. of multiplication of cosets}}{=} D^{G/N}(g_1 N g_2 N)$

$= D^{G/N}(g_1 N) D^{G/N}(g_2 N) = D^G(g_1) D^G(g_2)$

(2) Consider a faithful representation of G, $D \equiv D^G$, then

a) $g \mapsto D(g)^\dagger$

⇝ $D(g_1 g_2)^\dagger = (D(g_1) D(g_2))^\dagger = D(g_2)^\dagger D(g_1)^\dagger$

$\neq D(g_1)^\dagger D(g_2)^\dagger$ as not abelian

More formal way? Why show $O(g) := D(g)^\dagger$, $O(g_1 g_2) \overset{!}{=} O(g_1) O(g_2)$ ⇝ define new $D'(g) = D(g)^\dagger$

⟹ no representation

b) $g \mapsto (D(g^{-1}))^\dagger$

⇝ $D((g_1 g_2)^{-1})^\dagger = D(g_2^{-1} g_1^{-1})^\dagger = (D(g_2^{-1}) D(g_1^{-1}))^\dagger$

$= D(g_1^{-1})^\dagger D(g_2^{-1})^\dagger$

⟹ is a representation

c) $g \mapsto \det(D(g))$

⇝ $\det(D(g_1 g_2)) = \det(D(g_1) D(g_2)) = \det(D(g_1)) \det(D(g_2))$

⟹ is a representation

d) $g \mapsto \mathrm{Tr}(D(g))$

⇝ $\mathrm{Tr}(D(g_1 g_2)) = \mathrm{Tr}(D(g_1) D(g_2)) \overset{\text{in gen.}}{\neq} \mathrm{Tr}(D(g_1)) \mathrm{Tr}(D(g_2))$

⟹ no representation

A5.2 Define $\langle v,w\rangle := \frac{1}{|G|}\sum_{g\in G}\langle D^G(g)v, D^G(g)w\rangle_0$, with $\langle .,.\rangle_0$ Scalar Product

Need to show: • $\langle u,v\rangle = \langle v,u\rangle^*$, • $\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$

• $\langle u,\alpha v\rangle = \alpha\langle u,v\rangle$, • $\langle u,u\rangle > 0\ \forall u\neq 0$, • $\langle u,u\rangle = 0$ iff $u=0$ ✓

Why define different Scalar Product? we use it later in lecture as "group"? product

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• $$\langle v,w\rangle = \frac{1}{|G|}\sum_{g\in G}\langle D^G(g)v, D^G(g)w\rangle_0$$

$$\overset{\langle .,.\rangle SP}{=} \frac{1}{|G|}\sum_{g\in G}\langle D^G(g)w, D^G(g)v\rangle_0^*$$

$$= \left(\frac{1}{|G|}\sum_{g\in G}\langle D^G(g)w, D^G(g)v\rangle_0\right)^* = \langle w,v\rangle^*$$

Pull α out of $D^G(g)$?

• $$\langle u,v+w\rangle = \frac{1}{|G|}\sum_{g\in G}\langle D^G(g)u, D^G(g)(v+w)\rangle_0$$

$$= \frac{1}{|G|}\sum_{g\in G}\left(\langle D^G(g)u, D^G(g)v\rangle_0 + \langle D^G(g)u, D^G(g)w\rangle_0\right)$$

$$= \langle u,v\rangle + \langle u,w\rangle$$

• $$\langle u,\alpha v\rangle = \frac{1}{|G|}\sum_{g\in G}\langle D^G(g)u, D^G(g)(\alpha v)\rangle_0$$

$$= \frac{1}{|G|}\sum_{g\in G}\alpha\langle D^G(g)u, D^G(g)v\rangle_0 = \alpha\langle u,v\rangle$$

• $$\langle u,u\rangle = \frac{1}{|G|}\sum_{g\in G}\underbrace{\langle D^G(g)u, D^G(g)u\rangle_0}_{\substack{>0 \text{ for } D^G(g)u\neq 0\\ =0 \text{ for } D^G(g)u=0}}$$

$$\begin{cases} >0 & \text{for } u\neq 0\\ =0 & \text{for } u=0\end{cases}$$

$D^G(g)$ Conserves group structure (unique inverse element)

$D^G(g)v = 0\ \forall g\in G \Rightarrow v = D^{-1}(g)0 = 0$

If $D^G(g)$ Unitary:

$$\langle v,w\rangle = \frac{1}{|G|}\sum_{g\in G}\langle D^G(g)v, D^G(g)w\rangle_0 = \frac{1}{|G|}\sum_{g\in G}\langle D^G(g)^\dagger D^G(g)v, w\rangle_0$$

$$= \frac{1}{|G|}\sum_{g\in G}\langle v,w\rangle_0 = \frac{|G|}{|G|}\langle v,w\rangle_0 = \langle v,w\rangle_0$$

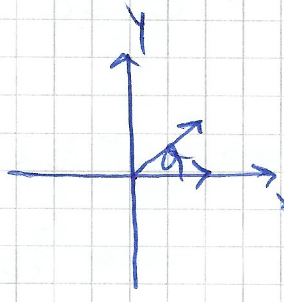
A 5.3

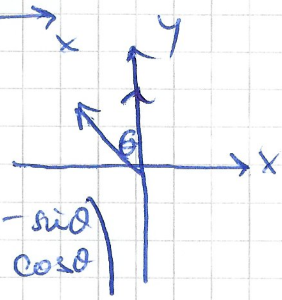
(1) $D_3 = \langle c, b \mid c^3 = e,\ b^2 = e,\ (bc)^2 = e \rangle$

We need rotations by $\theta = 120°$ and a reflection at some plane (let's say the x-axis)

$$\rightsquigarrow \begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$





$$\rightsquigarrow \begin{pmatrix}x\\y\end{pmatrix} \mapsto R\begin{pmatrix}x\\y\end{pmatrix} \text{ with } R = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

What exactly do we further have to prove? ⇝ construct like this

$\rightsquigarrow D(c) = R$

$D(c\cdot c) = R^2 = D(c)D(c)$ (trigonometry)

$D(c\cdot c\cdot c) = \mathbb{1}$

$$D(c) = \begin{pmatrix}-1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x\\-y\end{pmatrix} = \overbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}^{M}\begin{pmatrix}x\\y\end{pmatrix}$$

$$D(c^2) = \begin{pmatrix}-1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & -1/2\end{pmatrix}$$

$D(b) = M$

$D(b\cdot b) = M^2 = \mathbb{1}$

$$D(bc) := D(b)D(c) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

$$= \begin{pmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{pmatrix}, \quad D(bc) = \begin{pmatrix}-1/2 & -\sqrt{3}/2\\ -\sqrt{3}/2 & 1/2\end{pmatrix}$$

$$D(bc^2) = D(b)D(c)D(c) = MR^2$$

$$= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}, \quad D(bc^2) = \begin{pmatrix}-1/2 & \sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}$$

$$= \begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ -\sin 2\theta & -\cos 2\theta\end{pmatrix}$$

$\chi(c) = \mathrm{Tr}(D(c)) = -1,\ \chi(c^2) = -1,\ \chi(e) = 2$

$\chi(b) = 0,\ \chi(bc) = 0,\ \chi(bc^2) = 0.$

The conjugacy classes:

$\tilde{c} = gcg^{-1}$ for $g = c^u$ or $g = bc^u$

$\rightarrow c^u c c^{-u} = c \Rightarrow c$ itself

$\rightarrow bc^u c c^{-u} b = bcb = c^{-1} = c^2$

$\Rightarrow \{c, c^2\}$

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gbg^{-1}: $\rightarrow c^u b c^{-u} = bc^{3-u} c^{-u} = bc^{3-2u} = bc^{-2u}$

$\rightsquigarrow bc$

$\rightarrow bc^u b c^{-u} b = c^{3-u} c^{-u} b = bc^{3+2u} = bc^{2u}$

$\Rightarrow \{b, bc, bc^2\}$