Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

28.11.2017 **Group Theory Exercise 6**

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A6.1

Group G, D^G a d-dimensional representation and $\{v_1, \dots, v_d\}$ a basis of the vector space. M a nonsingular $d\times d$ matrix.

a) Subspace $V \subset L$ spanned by $\{v_1, \dots, v_{d_1}\}$, $d_1 < d$ invariant under D^G, say

$$D^G(g)(\lambda_1 v_1 + \dots + \lambda_{d_1} v_{d_1}) = \mu_1 v_1 + \dots + \mu_{d_1} v_{d_1}$$

Consider the equivalent representation $D^{G'} = M^{-1} D^G M$. Then $\{w_1, \dots, w_{d_1}\}$ is a basis of the vectorspace V', which is invariant under $D^{G'}$, where $M w_i \overset{!}{=} v_i \Leftrightarrow w_i = M^{-1} v_i$

$$D^{G'}(g)\,(\kappa_1 w_1 + \dots + \kappa_{d_1} w_{d_1})$$
$$= M^{-1} D^G(g) M (\kappa_1 w_1 + \dots + \kappa_{d_1} w_{d_1})$$
$$= M^{-1} D^G(g) (\kappa_1 v_1 + \dots + \kappa_{d_1} v_{d_1})$$
$$= M^{-1} (\xi_1 v_1 + \dots + \xi_{d_1} v_{d_1})$$
$$= \xi_1 w_1 + \dots + \xi_{d_1} w_{d_1}$$

b) let $v \in L$, $v = \sum_i a_i v_i$ and $w_i = M(v_i) = \sum_{j=1}^d M_{ji} v_j$

$\Rightarrow v_i = M^{-1}(w_i) = \sum_{\ell=1}^d M^{-1}_{\ell i} w_i$

representation $D^G(g)$ in basis $\{v_1, \dots, v_d\}$ of V:

$$D^G(g)(v_i) = \sum_{k=1}^d (D^G(g))_{ki} v_k$$

$$\Rightarrow D^G(g)(w_i) = D^G(g)\left(\sum_{j=1}^d M_{ji} v_j\right) \overset{M_{ji}\text{ numbers, } D \text{ linear}}{=} \sum_{j=1}^d M_{ji} D^G(g)(v_j)$$

$$= \sum_{j=1}^d M_{ji} \sum_{k=1}^d (D^G(g))_{kj} v_k = \sum_{j=1}^d \sum_{k=1}^d \sum_{\ell=1}^d \underbrace{M_{ji} (D^G(g))_{kj} M^{-1}_{\ell k}}_{M^{-1}_{\ell k} D^G(g)_{kj} M_{ji} = (\tilde{D}^G g)} w_\ell$$

$$= \sum_{\ell=1}^d (\tilde{D}^G(g))_{\ell i} w_\ell, \quad \tilde{D}^G(g) = M^{-1} D^G(g) M$$