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Group theory Exercise 7

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A7.1

$$D^{(j)}_{m'm}(\alpha,\beta,\gamma) = \langle j, m' | e^{-i/\hbar\, j_z \alpha} e^{-i/\hbar\, j_y \beta} e^{-i/\hbar\, j_z \gamma} | j m \rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \langle j m' | e^{-i/\hbar\, j_y \beta} | j m \rangle \quad (*)$$

$$\equiv e^{-i(m'\alpha + m\gamma)} d^{(j)}_{m'm}(\beta)$$

(1)

Consider $\langle j' m' | [j_z, T^{(\nu)}_n] | j m \rangle = \langle j' m' | \hbar n T^{(\nu)}_n | j m \rangle$

$$\Leftrightarrow \langle j' m' | j_z T^{(\nu)}_n - T^{(\nu)}_n j_z | j m \rangle = \hbar n \langle j' m' | T^{(\nu)}_n | j m \rangle$$

$$\Leftrightarrow \hbar (m' - m) \langle j' m' | T^{(\nu)}_n | j m \rangle = \hbar n \langle j' m' | T^{(\nu)}_n | j m \rangle$$

$$\rightsquigarrow n = m' - m \Leftrightarrow m' = n + m$$

If it's not fulfilled, $\langle j' m' | T^{(\nu)}_n | j m \rangle = 0$ in order to fulfill the upper equation.

(2)

In the lecture, we had

Where is this CG series coming from? → proven in lecture?! It's just a basis trafo

$$| j m \rangle = \sum_{m_1, m_2} | j_1 m_1 \rangle | j_2 m_2 \rangle \langle j_1 m_1 ; j_2 m_2 | j m \rangle$$

$$\langle j, m' | = \sum_{m_1', m_2'} \langle j_1 m_1' | \langle j_2 m_2' | \langle j m' | j_1 m_1' ; j_2 m_2' \rangle$$

And how to get the CG coeff. w/o looking into the table? → is just given by coeff. of basis trafo

$$\rightsquigarrow (*) = e^{-i(m'\alpha + m\gamma)} \sum_{\substack{m_1, m_2 \\ m_1', m_2'}} \langle j_1 m_1' | \langle j_2 m_2' | \langle j m' | j_1 m_1' ; j_2 m_2' \rangle e^{-\frac{i}{\hbar}(j_{1y} + j_{2y}}$$

$$\times | j_1 m_1 \rangle | j_2 m_2 \rangle \langle j_1 m_1 ; j_2 m_2 | j m \rangle$$

Why split $\langle j_2 m_2' | e^{\ldots} | j_1 m_1 \rangle$ like this? Actually would have to change order of $| j_1 m_1 \rangle | j_2 m_2 \rangle$?

triplet singlet

$$= e^{-i(m'\alpha + m\gamma)} \sum_{\substack{m_1, m_2 \\ m_1', m_2'}} \langle j m' | j_1 m_1' ; j_2 m_2' \rangle \langle j_1 m_1 ; j_2 m_2 | j m \rangle d^{(j_1)}_{m_1' m_1}(\beta) d^{(j_2)}_{m_2' m_2}(\beta)$$

(3) $d^{(1/2)}_{\pm 1/2 \pm 1/2}(\beta) = \cos(\beta/2)$, $d^{(1/2)}_{\pm 1/2 \mp 1/2}(\beta) = \mp \sin(\beta/2)$

(4)

$$d^{(j)}_{m'm} = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle j m' | j_1 m_1'; j_2 m_2' \rangle \langle j_1 m_1; j_2 m_2 | j m \rangle d^{(j_1)}_{m_1' m_1} d^{(j_2)}_{m_2' m_2}$$

$$d^{(1)}_{11} = \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \langle 1 1 | 1/2\, m_1'; 1/2\, m_2' \rangle \langle 1/2\, m_1; 1/2\, m_2 | 1 1 \rangle d^{(1/2)}_{m_1' m_1} d^{(1/2)}_{m_2' m_2}$$

$$= \left(\langle 1 1 | 1/2\, 1/2; 1/2\, 1/2 \rangle \right)^2 \left(d^{(1/2)}_{1/2\, 1/2} \right)^2 = \cos^2(\beta/2) = \frac{1}{2}(1 + \cos\beta)$$

How are j_1 and j_2 fixed? Can choose any comb. of them yielding j

A7.2

G a finite group and $D^{(\nu)}$ an n_ν-dim. irrep. of G

Consider $D_i^{(\nu)} = \sum_{g \in Cl_i} D^{(\nu)}(g)$

(1) $\forall h \in G: D^{(\nu)}(h) D_i^{(\nu)} (D^{\nu}(h))^{-1} = D^{(\nu)}(h) \sum_{g \in Cl_i} D^{(\nu)}(g) (D^{(\nu)}(h))^{-1}$

$= \sum_{g \in Cl_i} D^{(\nu)}(h) D^{(\nu)}(g) D^{(\nu)}(h^{-1}) = \sum_{g \in Cl_i} D^{(\nu)}(hgh^{-1})$

$= \sum_{g' \in Cl_i} D^{(\nu)}(g') = D_i^{(\nu)}$

Some elements might appear twice in hgh^{-1}?
$\rightsquigarrow hg_1h^{-1} = hg_2h^{-1}$
$\rightsquigarrow g_1 = g_2$

$\rightsquigarrow D^{(\nu)}(h) D_i^{(\nu)} = D_i^{(\nu)} D^{(\nu)}(h) \rightsquigarrow$ apply Schurs 2nd Lemma

$\rightsquigarrow D_i^{(\nu)} = \lambda_i \mathbb{1}_{n_\nu \times n_\nu}$

(2) $Tr(D_i^{\nu}) = Tr(\lambda_i \mathbb{1}_{n_\nu \times n_\nu}) = n_\nu \lambda_i$

$Tr(D_i^{\nu}) = Tr\left(\sum_{g \in Cl_i} D^{\nu}(g)\right) = \sum_{g \in Cl_i} Tr(D^{\nu}(g))$

$= |Cl_i| \, Tr(D^{\nu}(g))$ as $Tr(D^{\nu}(g)) = Tr(D^{\nu}(\tilde{g}))$
if g and $\tilde{g}$ conjugate elements

$\rightsquigarrow \lambda_i = \frac{|Cl_i| \, \chi(Cl_i)}{n_\nu}$

$\Leftrightarrow \lambda_i^2 = \frac{|Cl_i|^2 (\chi(Cl_i))^2}{n_\nu^2}$

$\Leftrightarrow \frac{n_\nu^2 \lambda_i^2}{|Cl_i|} = |Cl_i| \chi^2(Cl_i)$

$\Leftrightarrow \sum_{\text{class } i} \frac{n_\nu^2 \lambda_i^2}{|Cl_i|} = \sum_{\text{class } i} |Cl_i| \chi^2(Cl_i) = |G|$

$\Leftrightarrow n_\nu^2 = \frac{|G|}{\sum_{\text{class } i} \frac{\lambda_i^2}{|Cl_i|}}$

A 7.3

G a group, D^μ an n_μ-dim. irrep. of G

We have $\frac{1}{|G|} \sum_{g \in G} \chi^\mu(g) [\chi^\nu(g)]^* = \delta^{\mu\nu}$ from the lecture

This is some kind of complex scalar product w/ the vectors $\begin{pmatrix} \chi^\mu(g_1) \\ \vdots \\ \chi^\mu(g_{|G|}) \end{pmatrix}$, which are orthogonal given the above theorem and span a $|G|$-dimensional vectorspace

We can rewrite the above theorem as

$$\frac{1}{|G|} \sum_{\text{classes } Cl_i} N_i \chi^\mu(Cl_i) [\chi^\nu(Cl_i)]^* = \delta^{\mu\nu},$$ where N_i denotes the number of elements in Cl_i

Here, $(\mu = 1, \ldots, y)$ $\begin{pmatrix} \sqrt{N_1}\, \chi^\mu(Cl_1) \\ \vdots \\ \sqrt{N_{|G|}}\, \chi^\mu(Cl_{|G|}) \end{pmatrix}$ form a complex scalar product of a $|Cl|$ ($\hat{=}$ no's of conjugacy classes) - dim. vector space. They are orthogonal and thus can only be lower or equal to the number of conjugacy classes $|Cl|$, as this is the dimension of the vector space they span. Otherwise, we would have $|Cl|+1$ orthogonal vectors in a $|Cl|$-dim. vector space. ↯ $\rightsquigarrow y \leq |Cl|$