Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:

https://www.physics-and-stuff.com/

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A8.4

(1) \[ D_6 = \langle c, b | c^6 = e, b^2 = e, (bc)^2 = c^{-2} \]
\[ = \{ e, c, c^2, c^3, c^4, c^5, b, bc, bc^2, bc^3, bc^4, bc^5 \} \]

For the conjugacy classes, we notice:

\[ [c^i] = \{ gc^i g^{-1} | g \in D_6 \} \]
- \( bc^ib = c^{n-i} \)
- \( bc^k c (bc)^{-1} = bc^k c c^{n-k} b = bc^{n+i} b = c^{-i} \)

\[ [bc^i] = \{ gb e e g^{-1} | g \in D_6 \} \]
- \( b b c^i b = b c^{n-i} \)
- \( c^k b c^k = b c^{n-k} c^k b = b c^{n+i} \)
- \( (bc^k)(bc^i)(bc^k)^{-1} = b c^k b c^{n-k} b = b c^{n+i} c^{n-k} b = c^{n-k} c b c^{n-i} c = bc^{2k-i} \)

\[ [e] = \{ e \} \quad [b] = \{ b, b c, b c^2, b c^4 \} \]
\[ [c] = \{ c, c^5 \} \quad [bc] = \{ bc, b c^3, b c^5 \} \]
\[ [c^2] = \{ c^2, c^4 \} \]
\[ [c^3] = \{ c^3 \} \]

(2) For the number of irreducible representations, we use:

\[ N_{irreps} = N_{conjugacy \ classes} = 6 \]

For the dimension, we use:

\[ N_D = 12 + (1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2) \]
\[ = 12 - d_2^2 + d_2^2 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \]
\[ \Rightarrow 2 \times 2 \text{dimensional reps and } 4 \times 1 \text{dimensional reps} \]
The first row and column can easily be filled.

Orthogonality Theorems:
- \( N_{\alpha} \), \( \delta_{\mu} = \sum_{\nu=1}^{6} X^\nu (\text{cell}) (X^\nu (\text{cell}))^* \)
- \( N_{\alpha} \delta_{\mu} = \sum_{\nu=1}^{6} N_{\nu} X^\nu (\text{cell}) (X^\nu (\text{cell}))^* \)

We have \( X(c^2) = X(c c^2) = (X(c))^4 \Rightarrow X(c^2) = 1 \)

As \( X(c^2) \neq 0 \), \( X(c^2) = X(c c^2) = (X(c))^4 \Rightarrow X(c^2) = \pm 1 \)

for all 1 dim reps.

\[ \Rightarrow 6 \stackrel{1}{\sim} 1^2 + 1^2 + 1^2 + 1^2 + x^2 + y^2 \Rightarrow x, y = \pm 1 \]

\( X(c) = X(b b) = X(b) X(b) \Rightarrow X(b) = \pm 1 \) for 1 dim,

\[ \Rightarrow 4 \stackrel{1}{\sim} x^2 + y^2 = 0 \]

Analogously for \( X(c) = X(b c) X(b c) \)

For \( X(c) = X(c c^3) X(c c^3) \Rightarrow X(c c^3) = \pm 1 \)

\[ 6 \stackrel{3}{\sim} a^2 + 3 (\pm 1)^2 + a^2 + b^2 \Rightarrow a, b = \pm 2 \]

\( X(c) = X(c c^3) = X(c) X(c) \xrightarrow{\text{conj}} X(c) X(c) \Rightarrow X(c) = \pm 1 \)

\[ 6 \xrightarrow{1} a^2 + 3 (\pm 1)^2 + c^2 + d^2 \Rightarrow c, d = \pm 1 \]

\( \Rightarrow \) Table above.
AS \[ 0 = 1 + 1 + 1 + 1 + 2x + 2y \quad \Rightarrow \quad x = y = -1 \]
\[ 0 = 1 + 2 + 4 ( \pm 1 ) + 2 ( \pm 1 ) + 3 ( \pm 1 ) + 3 ( \pm 1 ) \]

\[ \Rightarrow 3 \text{ possibilities: } -4, -4, +4, -4, -4, +4, +4, +4, -4, -4 \]

\[ \begin{array}{ccccccc}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1' & 1 & -1 & -1 & 1 & -1 & -1 \\
1'' & 1 & -1 & -1 & 1 & -1 & -1 \\
1''' & 1 & 1 & 1 & 1 & -1 & -1 \\
2 & 2 & 2 & -1 & -1 & 0 & 0 \\
2' & 2 & -2 & 1 & -1 & 0 & 0 \\
\end{array} \]

\[ 0 = 2 + a + 2c - 2 \quad \text{w/} \quad c = \pm 1, a = \pm 2 \]

\[ \Rightarrow 2 \text{ solutions} \]

(4) \[ b = (6) (4) (2) (1) (3) \quad \Rightarrow \quad c = (1) (27) (45) (6) \]

\[ c^2 = (1) (27) (45) (6) \]

\[ c^3 = (1) (27) (45) (6) \]

\[ c^4 = (5) (3) (1) (6) \]

\[ c^5 = (6) (5) (4) (3) \]

\[ (bc) = (6) (12) (5) (3) \]

\[ (bc)^2 = (4) (5) (2) (1) \]

\[ (bc)^3 = (6) (5) (12) (3) \]

\[ (bc)^4 = (4) (5) (12) (3) \]

\[ (bc)^5 = (6) (5) (12) (3) \]

(5) For \( [b^7] = \{ b, bc, bc^2, bc^4 \} \), there are 2 atoms left at their positions, for the others none.
6) Looking at \( \begin{bmatrix} \bullet & (i) \end{bmatrix} \begin{bmatrix} (i) \end{bmatrix} \begin{bmatrix} (i) \end{bmatrix} \begin{bmatrix} (i) \end{bmatrix} \), there are only entries on the diagonals, if an element is unchanged.

We consider the 6-dim rep \( D^6 \) of \( D_6 \) (as above)

\( \chi^3(e) = 6 \), \( \chi^3(EB) = 2 \), \( \chi^3("\text{else"}) = 0 \)

{eqn}
\[ \text{Corr.} \quad \frac{1}{N_{D_6}} \sum_{g \in D_6} |\chi^3(g)|^2 = \frac{1}{12} (3 \cdot 2^2 + 6^2) = 47/1 \]
{eqn}

We notice reducible

\[ D^6 = \sum_{i=1}^{j} \mathbb{D}_i \otimes \chi_i \]

with \( i \), \( \mathbb{D}_i = \frac{1}{N_{D_6}} \sum_{g \in D_6} \mathbb{D}_{i}^g \sum_{g \in D_6} \chi_i(g)^* \chi_i(g) \)

\[ q_1 = \frac{1}{12} (3 \cdot 2 \cdot 4 + 1 \cdot 4 \cdot 6) = 1 \]

\[ q_{11} = \frac{1}{12} (3 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 6) = 4 \]

\[ q_{41} = \frac{1}{12} (3 \cdot 2 \cdot (-1) + 1 \cdot 1 \cdot 6) = 0 \]

\[ q_{14} = \frac{1}{12} (3 \cdot 2 \cdot (-1) + 1 \cdot 1 \cdot 6) = 0 \]

\[ q_2 = \frac{1}{12} (3 \cdot 2 \cdot 0 + 1 \cdot 2 \cdot 6) = 4 \]

\[ q_2' = \frac{1}{12} (3 \cdot 2 \cdot 0 + 1 \cdot 2 \cdot 6) = 4 \]

\[ D^6 = q_1 + q_{11} + q_2 + q_2' \]

8) We have 4 states: 2 singlets and 2 doublets.

\( \text{Why can we say } \)

\( \text{What about } \)

\( \text{Physics/Fields. } \)

\( \text{How are we } \)

\( \text{Working with? } \)

\( \text{Do Hamiltonian } \)

\( \text{correlated? } \)