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<https://www.physics-and-stuff.com/>

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Group theory 8th Exercise

Mami Zake

AS.1

$$(1) D_6 = \langle c, b \mid c^6 = e, b^2 = e, (bc)^2 = e \rangle \\ = \{e, c, c^2, c^3, c^4, c^5, b, bc, bc^2, bc^3, bc^4, bc^5\}$$

through to  
check Conj.  
Classes w.r.t  
the generators?

For the conjugacy classes, we notice,

$$[c^i] = \{g c^i g^{-1} \mid g \in D_6\}$$

$$- b c^i b = c^{n-i}$$

$$- b c^k c^i (b c^k)^{-1} = b c^k c^i c^{n-k} b = b c^{n+i} b = c^{-i}$$

$$[bc^i] = \{g bc^i g^{-1} \mid g \in D_6\}$$

$$- b bc^i b = bc^{n-i}$$

$$- c^k bc^i c^k = bc^{n-k+i+k} = bc^{n+i}$$

$$- (bc^k)(bc^i)(bc^k)^{-1} = bc^k bc^i c^{n-k} b = c^{n-k} c^i c^{n-k} b$$

$$= c^{n-k} c^i b c^k = c^{n-k} bc^{n-i+k} = bc^k c^{n+i-k} = bc^{2k-i}$$

$$\Rightarrow [e] = \{e\}$$

$$[b] = \{b, bc^2, bc^4\}$$

$$[c] = \{c, c^5\}$$

$$[bc] = \{bc, bc^3, bc^5\}$$

$$[c^2] = \{c^2, c^4\}$$

$$[c^3] = \{c^3\}$$

(2) For the number of irreducible representations, we use

$$N_{\text{irrep}} = N_{\text{cl}} = 6$$

For the dimensions, we use

$$N_{\text{cl}} = |X^1(e)|^2 + \dots + |X^6(e)|^2$$

$$\Rightarrow 12 = d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2$$

$\Rightarrow 2 \times 2$  dimensional irreps and  $4 \times 1$  dimensional irreps

(3)

$D_6$	$[e]$	$\overset{1}{c_2}$ $[c^2]$	$\overset{2}{c_0}$ $[c]$	$\overset{2}{c_3}$ $[c^2]$	$\overset{3}{c_2}$ $[b]$	$\overset{3}{c_2}$ $[bc]$
1	1	1	1	1	1	1
1'	1	$\pm 1$	$\pm 1$	1	$\pm 1$	$\pm 1$
1''	1	$\pm 1$	$\pm 1$	1	$\pm 1$	$\pm 1$
1'''	1	$\pm 1$	$\pm 1$	1	$\pm 1$	$\pm 1$
2	2	$\overset{a}{\pm 2}$	$\overset{a}{\pm 1}$	$\overset{x}{\pm 1}$	$\overset{w}{0}$	0
2'	2	$\overset{b}{\pm 2}$	$\overset{d}{\pm 1}$	$\overset{y}{\pm 1}$	$\overset{z}{0}$	0

Why only integer numbers?

The first row and column can easily be filled

Orthogonality theorems:

$$\frac{N_j}{N_i} \delta_{ij} = \sum_{\nu=1}^6 \chi^\nu(c_i) (\chi^\nu(c_j))^*$$

$$N_i \delta_{ij} = \sum_{\nu=1}^6 N_\nu \chi^\nu(c_i) (\chi^\nu(c_j))^*$$

We have  $\chi(c^1) = \chi(c^4) = (\chi(c^1))^2 \Rightarrow \chi(c^2) = 1$

as  $\chi(c^2) \neq 0$ ,  $\chi(c^2) = \chi(c^4) = (\chi(c^2))^{-1} \Rightarrow \chi(c^2) = \pm 1$   
for all 1 dim. irrep.

$$\Rightarrow 6 \stackrel{!}{=} 1^2 + 1^2 + 1^2 + 1^2 + x^2 + y^2 \Rightarrow x, y = \pm 1$$

$\chi(e) = \chi(b \cdot b) = \chi(b)\chi(b) \Rightarrow \chi(b) = \pm 1$  for 1 dim.

$$\Rightarrow 4 = 1^2 + 3(\pm 1)^2 + w^2 + z^2 \Rightarrow w, z = 0$$

analogously for  $\chi(e) = \chi(bc) \chi(bc)$

For  $\chi(e) = \chi(c^3) \chi(c^3) \Rightarrow \chi(c^3) = \pm 1$

$$12 \stackrel{!}{=} 1^2 + 3(\pm 1)^2 + a^2 + b^2 \Rightarrow a, b = \pm 2$$

$\chi(e) = \chi(c \cdot c^5) = \chi(c) \chi(c^5) \stackrel{\text{same conj.}}{=} \chi(c) \chi(c) \Rightarrow \chi(c) = \pm 1$

$$6 \stackrel{!}{=} 1^2 + 3(\pm 1)^2 + c^2 + d^2 \Rightarrow c, d = \pm 1$$

$\Rightarrow$  table above

As the 3 \* 1 dim. irreps are int. changeable, we just have to take care for the orthogonality theorems to be fulfilled

• A)  $0 = 1 + 1 + 1 + 1 + 2x + 2y \Rightarrow x = y = -1$

•  $0 = 1 + 2 + (\pm 1) + 2(\pm 1) + 3(\pm 1) + 3(\pm 1)$

$\Rightarrow$  3 possibilities:  $-1, -1, +1, -1$   
 $-1, -1, -1, +1$   
 $+1, +1, -1, -1$

really only 3 possibilities?

$D_6$	$[e]$	$[c^1]$	$[c^2]$	$[c^3]$	$[b^3]$	$[bc^3]$
1	1	1	1	1	1	1
1'	1	-1	-1	1	1	-1
1''	1	-1	-1	1	-1	1
1'''	1	1	1	1	-1	-1
2	2	2	-1	-1	0	0
2'	2	-2	1	-1	0	0

•  $0 = 2 + a + 2c - 2 \quad w / c = \pm 1, a = \pm 2$

$\Rightarrow$  2 solutions

(4)  $b = (6)(15)(24)(3) \quad , \quad c = (123456)$

$\Rightarrow c^2 = (123456)^2 = (135)(246)$

$c^3 = (14)(25)(36)$

$c^4 = (531)(642)$

$c^5 = (654321)$

$(bc) = (6)(15)(24)(3)(123456) = (41)(23)(56)$

$(bc^2) = (13)(5)(2)(46)$

$(bc^3) = (12)(36)(45)$

$(bc^4) = (53)(11)(62)(4)$

$(bc^5) = (61)(25)(34)$

(5) For  $[b]_c = \{b, bc^2, bc^4\}$ , there are 2 atoms left at their positions, for the others none.

6) Looking at  $\underbrace{\begin{pmatrix} * & & * \\ | & & | \\ * & & * \end{pmatrix}}_6 \begin{pmatrix} * \\ | \\ * \end{pmatrix}$ , there are only entries on the diagonals, if an element is unchanged.

↳ consider the 6-dim rep  $D^B$  of  $D_6$  (as above)

↳  $\chi^B(e) = 6$ ,  $\chi^B([b]) = 2$ ,  $\chi^B(\text{"else"}) = 0$

7) Consider  $\frac{1}{N_{D_6}} \sum_{g \in D_6} |\chi^B(g)|^2 = \frac{1}{12} (3 \cdot 2^2 + 6^2) = 4 > 1$

↳ it is reducible

$D^B = \sum_{\chi} q_{\chi} \chi$  w/  $q_{\chi} = \frac{1}{N_{D_6}} \sum_{i=1}^{N_{D_6}} N_i (\chi^B(a_i))^* \chi(a_i)$

$q_1 = \frac{1}{12} (3 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 6) = 1$

$q_{1'} = \frac{1}{12} (3 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 6) = 1$

$q_{1''} = \frac{1}{12} (3 \cdot 2 \cdot (-1) + 1 \cdot 1 \cdot 6) = 0$

$q_{2''} = \frac{1}{12} (3 \cdot 2 \cdot (-1) + 1 \cdot 1 \cdot 6) = 0$

$q_2 = \frac{1}{12} (3 \cdot 2 \cdot 0 + 1 \cdot 2 \cdot 6) = 1$

$q_{2'} = \frac{1}{12} (3 \cdot 2 \cdot 0 + 1 \cdot 2 \cdot 6) = 1$

↳  $D^B = q_1 + q_{1'} + q_2 + q_{2'}$

8) We have 4 states; 2 singlets and 2 doublets

Why can we say sth. about physics/spectrum if there are just 4 states w/ mult? No Hamiltonian considered?