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The Gaussian Integral with Offset and Complex Coefficients

An Attempt for a Rigorous Derivation

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Abstract

In this article, we want to present a semi-rigorous proof of

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-ax^2+bx} = \mathrm{e}^{b^2/(4a)} \sqrt{\frac{\pi}{a}}, \qquad a, b \in \mathbb{C}, \quad \mathrm{Re}(a) \geq 0,$$
$$\mathrm{Re}(a) = 0 \implies \mathrm{Im}(a) \neq 0 \wedge \mathrm{Re}(b) = 0,$$

where the integral on the left-hand side is also known as a 'Gaussian Integral'. In the lecture, we were a bit careless in extending the validity of this formula from $a \in \mathbb{R}^+$, $b = 0$ to $a \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, $b = 0$, then to $a, b \in \mathbb{C}$, $\mathrm{Re}(a) > 0$, and ultimately to $a, b \in \mathbb{C}$, $\mathrm{Re}(a) \geq 0$, and we will see that it is in fact not trivial at all to extend the formula to the complex plane. We want to emphasize that the result is well known and our proof will be based on ideas from many different references, in particular Refs. [1, 2, 3, 4, 5, 6, 7], that is much of this article will be the intellectual property of other authors — we want to apologize to the authors for not citing these references properly throughout this work. However, we do not only collect the arguments of said references in a clear and comprehensible manner here but we also supplement them by some own input and explanations. Moreover, we will make use of certain results (that is e.g. theorems) from the Refs. [8, 9, 10, 11, 12], which we will also cite appropriately where needed. The Refs. [13, 14] might be interesting to the reader as well but are beyond the scope of this work; one of these references is an open question (at the time of writing) on StackExchange that will be referred to again in this article. It is likely that Mathematicians will not consider our proof to be rigorous — which is why we call it a semi-rigorous proof in the first place — since we might be a little bit sloppy regarding the validity of certain steps. For the reader who wants to learn more on Gaussian integrals and complex analysis, we refer to the above references.

1 The Gaussian Integral

It is the ultimate goal of this article to prove the following theorem, where we will proceed in small steps by proving several minor results.

Theorem 1. *Let* $a, b \in \mathbb{C}$ *with* $\mathrm{Re}(a) \geq 0$*, where* $\mathrm{Re}(a) = 0$ *requires* $\mathrm{Im}(a) \neq 0$ *and* $\mathrm{Re}(b) = 0$*. Then*

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} = \mathrm{e}^{b^2/(4a)} \sqrt{\frac{\pi}{a}}.$$

1.1 Real Coefficient without Offset

In a first step, we consider $b = 0$ and $a \in \mathbb{R}^+$, leading to the following Lemma.

Lemma 2. *Let* $a \in \mathbb{R}^+$*. Then*

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

Proof. We define

$$I(a) \coloneqq \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2},$$

where it is easy to see that the integral diverges for $a \in \mathbb{R}^- \cup \{0\}$, namely

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} \overset{\tilde{a}:=-a\geq 0}{=} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\tilde{a}x^2} \geq \int_{-\infty}^{\infty} \mathrm{d}x\, 1 \to \infty,$$

while for $a \in \mathbb{R}^+$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} &= 2\int_{0}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = 2\left[\int_{0}^{1} \mathrm{d}x\, \mathrm{e}^{-ax^2} + \int_{1}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2}\right] \\ &< 2\left[\int_{0}^{1} \mathrm{d}x\, \mathrm{e}^{-ax^2} + \int_{1}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax}\right] = 2\left[\underbrace{\int_{0}^{1} \mathrm{d}x\, \mathrm{e}^{-ax^2}}_{\text{finite}} - \frac{1}{a}\mathrm{e}^{-ax}\Big|_{1}^{\infty}\right] < \infty. \end{aligned}$$

Here, we used that the function $f(x) = \mathrm{e}^{-ax^2}$ is continuous (thus bounded) on the compact interval $I = [0, 1]$ and the fact that integrating such a function over that interval yields a finite result. In order to find the value of $I(a)$ for $a \in \mathbb{R}^+$, we now consider said expression squared, i.e.

$$
\begin{aligned}
I(a)I(a) &= \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-ay^2} \overset{\text{Fubini}}{=} \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, \mathrm{e}^{-a(x^2+y^2)} \\
&\underset{\text{coord.}}{\overset{\text{pol.}}{=}} \int_0^{\infty} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\;\, r\mathrm{e}^{-ar^2} = 2\pi \left[-\frac{1}{2a}\, \mathrm{e}^{-ar^2} \right] \Big|_0^{\infty} = \frac{\pi}{a},
\end{aligned}
$$

so that we readily deduce the value of $I(a)$ by taking the square root of this expression.[1] In calculating the above, we used Fubini's theorem and introduced polar coordinates [8, 9]. □

1.2 Complex Coefficient without Offset

We now want to extend the validity of the above result to the complex, that is $b = 0$ and $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$; more specifically, we have the following Corollary.

Corollary 3. *Let $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. Then*

$$
\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}.
$$

Proof. We start with the result of Lemma 2 and observe that the left-hand and right-hand side of that equation are analytic expressions of $a \in \mathbb{C}$ provided that $\mathrm{Re}(a) > 0$. Indeed, the left-hand side is analytic in this case since

$$
\left| \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} \right| \leq \int_{-\infty}^{\infty} \mathrm{d}x\, \left| \mathrm{e}^{-ax^2} \right| = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{Re}(a)x^2} < \infty
$$

for $\mathrm{Re}(a) > 0$, as shown before. Moreover, regarding the right-hand side, the function $g(a) = \sqrt{\pi}/\sqrt{a}$ is analytic for $a \in \mathbb{C} \setminus (-\infty, 0]$, choosing the principal square root function for evaluation; in order to see this, note that $h(a) = \sqrt{a}$ is analytic for $a \in \mathbb{C} \setminus (-\infty, 0]$ on the principal branch of the square root, whereas it has a branch cut for $a \in (-\infty, 0]$, and that the reciprocal of an analytic function is analytic where the function is analytic except for the region where the function is zero, i.e. $a = 0$, making the reciprocal singular.[2] Having argued that the left-hand and right-hand

[1] The idea of this proof goes back to Poisson [2]. There are many more ways to calculate the value of the integral $I(a)$, see e.g. Ref. [3].

[2] Unfortunately, I am not able to provide a proof for the divergence of $\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2}$, $a \in \mathbb{C}$, $\mathrm{Re}(a) < 0$; I opened a thread on StackExchange for this [14]. However, to some extent, it appears clear that this is the case because $\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-(\mathrm{Re}(a)+\mathrm{i}\mathrm{Im}(a))x^2}$, where $|\mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)x^2}| = 1$ is oscillating and $\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{Re}(a)x^2}$ diverges for $\mathrm{Re}(a) < 0$, as shown before.

side of the equation in Lemma 2 are analytic expressions of $a \in \mathbb{C}$ if $\mathrm{Re}(a) > 0$, we can use the identity theorem [10, 11, 12] to perform the analytic continuation of said equation to $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$. □

In a next step, we want to perform the analytic continuation of the formula to $b = 0$ and $a \in \mathbb{C}$ with $\mathrm{Re}(a) \geq 0$, where $\mathrm{Re}(a) = 0$ requires $\mathrm{Im}(a) \neq 0$, i.e. we want to also cover the case $\mathrm{Re}(a) = 0$. This is compiled in the following Lemma.

Lemma 4. *Let $a \in \mathbb{C}$ with $\mathrm{Re}(a) = 0$ and $\mathrm{Im}(a) \neq 0$. Then*

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \sqrt{\frac{\pi}{a}}.$$

Proof. We start by observing that $\mathrm{Im}(a) \neq 0$ is a necessary condition for $a \in \mathbb{C}$, $\mathrm{Re}(a) = 0$ because the integral would trivially diverge otherwise,

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2} \underset{\mathrm{Im}(a)=0}{\overset{\mathrm{Re}(a)=0}{=}} \int_{-\infty}^{\infty} \mathrm{d}x\, 1 \to \infty.$$

Although one might be tempted to prove the Lemma with the help of the identity theorem as well, the analyticity property of the left-hand side of the equation in Lemma 2 cannot be extended to $\mathrm{Re}(a) = 0$ as easy as it was the case for $\mathrm{Re}(a) > 0$; in particular, neither the left-hand nor the right-hand side of said equation are analytic for $\mathrm{Re}(a) = 0 = \mathrm{Im}(a)$.[3] Instead, we let $a \in \mathbb{C}$, $\mathrm{Re}(a) = 0$, $\mathrm{Im}(a) \neq 0$, where we assume $\mathrm{Im}(a) < 0$ w.l.o.g. and consider the closed contour integration

$$\oint_{\Gamma(m)} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)z^2} = 0,$$

where $\Gamma(m)$ is the integration contour depicted in Fig. 1, $m \in \mathbb{R}^+$ being some arbitrary constant, and the integration vanishes due to CAUCHY's theorem [10, 11] and the fact that the integrand $k(z) = \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)z^2}$ is an analytic function in the region enclosed by $\Gamma(m)$ and on its boundary. We split the integration contour into the three parts $\gamma_1(m)$, $\gamma_2(m)$, and $\gamma_3(m)$, so that

$$\oint_{\Gamma(m)} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)z^2} = \underbrace{\int_{\gamma_1(m)} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)z^2}}_{\gamma_1(m):\, z=x,\, x\in[0,m]} + \underbrace{\int_{\gamma_2(m)} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)z^2}}_{\gamma_2(m):\, z=m+\mathrm{i}y,\, y\in[0,m]} + \underbrace{\int_{\gamma_3(m)} \mathrm{d}z\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)z^2}}_{\gamma_3(m):\, z=x+\mathrm{i}x=\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}x,\, x\in[m,0]}$$

[3] Here, too, I am unable to provide a direct proof for the convergence of the integral for $a \in \mathbb{C}$, $\mathrm{Re}(a) = 0$, $\mathrm{Im}(a) \neq 0$, which is also part of my aforementioned question on StackExchange [14]. Nevertheless, we will explicitly calculate the integral and hence — a posteriori — see that it exists. Note that if we could prove the existence/analyticity beforehand, we were allowed to analytically continue to $\mathrm{Re}(a) = 0$, $\mathrm{Im}(a) \neq 0$ with the help of the identity theorem.

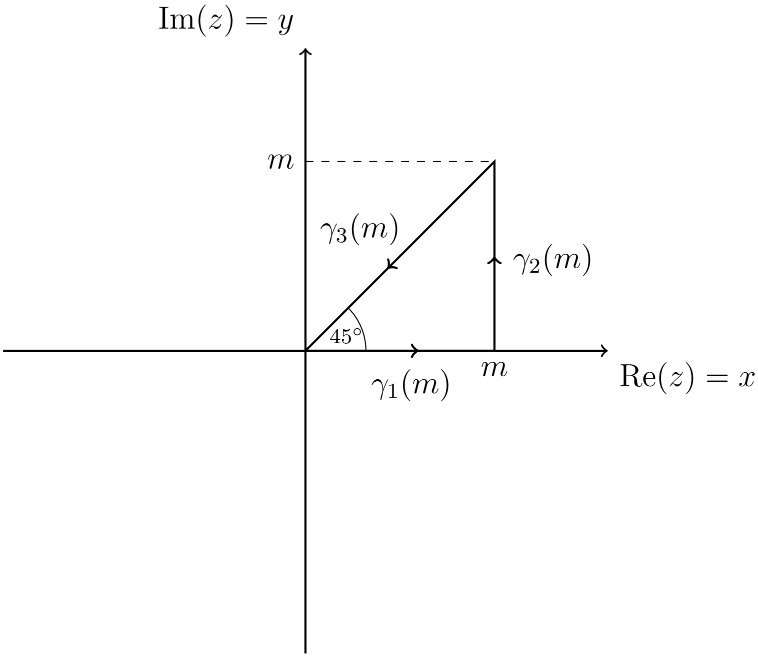

Figure 1: The closed integration contour $\Gamma(m)$ split into the three pieces $\gamma_1(m)$, $\gamma_2(m)$, and $\gamma_3(m)$, i.e. $\Gamma(m) = \gamma_1(m) \cup \gamma_2(m) \cup \gamma_3(m)$.

$$= \int_0^m \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)x^2} + \int_0^m \mathrm{d}y\, \mathrm{i}\mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)(m+\mathrm{i}y)^2} + \int_m^0 \mathrm{d}x\, \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)(2\mathrm{i})x^2}.$$

One readily deduces that the integration over $\gamma_2(m)$ vanishes in the limit $m \to \infty$, in particular

$$\left| \int_0^m \mathrm{d}y\, \mathrm{i}\mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)(m+\mathrm{i}y)^2} \right| \leq \int_0^m \mathrm{d}y\, \left| \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)(m+\mathrm{i}y)^2} \right| = \int_0^m \mathrm{d}y\, \mathrm{e}^{2\mathrm{Im}(a)my}$$
$$= \left[\frac{1}{2\mathrm{Im}(a)m} \mathrm{e}^{2\mathrm{Im}(a)my} \right] \Bigg|_0^m = \frac{\mathrm{e}^{2\mathrm{Im}(a)m^2} - 1}{2\mathrm{Im}(a)m} \underset{\mathrm{Im}(a)<0}{\overset{m\,\to\,\infty}{\longrightarrow}} 0.$$

Then, using the result of Cauchy's theorem from above, we find

$$\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)x^2} = -\sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4} \int_\infty^0 \mathrm{d}x\, \mathrm{e}^{2\mathrm{Im}(a)x^2} \overset{x \equiv \tilde{x} := \sqrt{2}x}{=} \mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{Im}(a)x^2}$$

in the limit $m \to \infty$ and thus

$$\int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)x^2} = 2\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}\mathrm{Im}(a)x^2} = 2\mathrm{e}^{\mathrm{i}\pi/4} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{Im}(a)x^2} = \mathrm{e}^{\mathrm{i}\pi/4} \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{Im}(a)x^2}$$
$$\overset{\mathrm{Im}(a)<0}{=} \sqrt{\mathrm{i}} \sqrt{\frac{\pi}{-\mathrm{Im}(a)}} = \sqrt{\frac{\pi}{\mathrm{i}\mathrm{Im}(a)}}.$$

where we chose the principal branch of the square root in order to evaluate $\mathrm{e}^{\mathrm{i}\pi/4}$.

By taking the complex conjugate of the second last equation, we find

$$\int_0^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{i}\mathrm{Im}(a)x^2} = \mathrm{e}^{-\mathrm{i}\pi/4} \int_0^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{Im}(a)x^2},$$

so that — again using the principal branch of the square root — we similarly obtain

$$\begin{aligned}\int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-(-\mathrm{i}\mathrm{Im}(a))x^2} &= \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{i}\mathrm{Im}(a)x^2} = \mathrm{e}^{-\mathrm{i}\pi/4} \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{Im}(a)x^2} \overset{\mathrm{Im}(a)<0}{=} \sqrt{-\mathrm{i}}\sqrt{\frac{\pi}{-\mathrm{Im}(a)}} \\ &= \sqrt{\frac{\pi}{-\mathrm{i}\mathrm{Im}(a)}}.\end{aligned}$$

Note that by the last and third from last equation we cover both cases of $\mathrm{Im}(a) \neq 0$ in Lemma 4, namely $\mathrm{Im}(a) > 0$ and $\mathrm{Im}(a) < 0$. □

1.3 Complex Coefficient with Offset

In a final step, we want to include a linear offset in the exponential function, i.e. $a, b \in \mathbb{C}$ with $\mathrm{Re}(a) \geq 0$ and where $\mathrm{Re}(a) = 0$ requires $\mathrm{Im}(a) \neq 0$ and $\mathrm{Re}(b) = 0$. This is what is compiled in Theorem 1 and will be proven in the following.

Proof of Theorem 1. We let $a, b \in \mathbb{C}$, $\mathrm{Re}(a) \geq 0$, where we require $\mathrm{Im}(a) \neq 0$ and $\mathrm{Re}(b) = 0$ if $\mathrm{Re}(a) = 0$, for otherwise the integral diverges due to Lemma 4 and

$$\begin{aligned}\int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} &= \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{i}\mathrm{Im}(b)x}, \quad \text{for } \mathrm{Im}(a) = 0, \mathrm{Re}(b) = 0,\\ \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} &= \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{Re}(b)x+\mathrm{i}\mathrm{Im}(b)x}, \quad \text{for } \mathrm{Im}(a) = 0, \mathrm{Re}(b) \neq 0,\\ \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} &= \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{\mathrm{Re}(b)x-\mathrm{i}(\mathrm{Im}(a)x^2-\mathrm{Im}(b)x)}, \quad \text{for } \mathrm{Im}(a) \neq 0, \mathrm{Re}(b) \neq 0,\end{aligned}$$

which give ill-defined expressions, the first of these being connected to a well-known representation of the Dirac delta distribution in Fourier analysis.[4] To continue with the proof, let us complete the square in the exponential function,

$$\begin{aligned}\int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} &= \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-a(x-b/(2a))^2+b^2/(4a)} = \mathrm{e}^{b^2/(4a)} \int_{-\infty}^\infty \mathrm{d}x\, \mathrm{e}^{-a(x-b/(2a))^2}\\ &\underset{\frac{b}{2a}=\mathrm{Re}(\frac{b}{2a})+\mathrm{i}\mathrm{Im}(\frac{b}{2a})}{\overset{x\equiv\hat{x}:=x-\frac{b}{2a}}{=}} \mathrm{e}^{b^2/(4a)} \int_{-\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})}^{\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})} \mathrm{d}x\, \mathrm{e}^{-ax^2}.\end{aligned}$$

[4] As the reader may have noticed, we are being more and more sloppy about anything relating to the convergence and divergence of the considered integrals, which is merely due to our incapability of providing rigorous proofs for this; as before, we refer the reader to Refs. [13, 14].

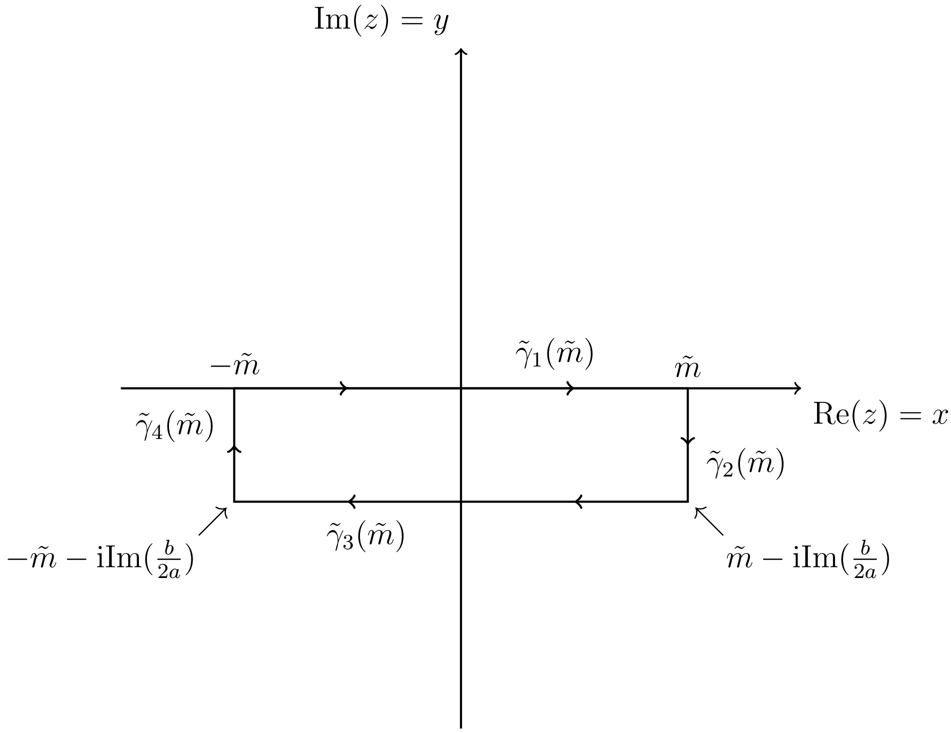

Figure 2: The closed integration contour $\tilde{\Gamma}(\tilde{m})$ split into the four pieces $\tilde{\gamma}_1(\tilde{m})$, $\tilde{\gamma}_2(\tilde{m})$, $\tilde{\gamma}_3(\tilde{m})$, and $\tilde{\gamma}_4(\tilde{m})$, i.e. $\tilde{\Gamma}(\tilde{m}) = \tilde{\gamma}_1(\tilde{m}) \cup \tilde{\gamma}_2(\tilde{m}) \cup \tilde{\gamma}_3(\tilde{m}) \cup \tilde{\gamma}_4(\tilde{m})$.

The case $\mathrm{Re}(a) = 0$, $\mathrm{Im}(a) \neq 0$, $\mathrm{Re}(b) = 0$ now follows immediately, given that it implies $\mathrm{Im}(b/(2a)) = 0$ and thus by Lemma 4

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} = \mathrm{e}^{b^2/(4a)} \int_{-\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})}^{\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})} \mathrm{d}x\, \mathrm{e}^{-ax^2} \overset{\mathrm{Re}(a)=0,\,\mathrm{Im}(a)\neq 0}{\underset{\mathrm{Re}(b)=0}{=}} \mathrm{e}^{b^2/(4a)} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2}$$
$$\overset{\mathrm{Re}(a)=0}{\underset{\mathrm{Im}(a)\neq 0}{=}} \sqrt{\frac{\pi}{a}}.$$

In order to calculate the integral in the second last equality for $\mathrm{Re}(a) > 0$, we now assume $\mathrm{Im}(b/(2a)) > 0$ w.l.o.g. (for $\mathrm{Im}(b/(2a)) < 0$, we merely have to mirror the integration contour at the x-axis) and consider the closed contour integration

$$\oint_{\tilde{\Gamma}(\tilde{m})} \mathrm{d}z\, \mathrm{e}^{-az^2} = 0,$$

where $\tilde{\Gamma}(\tilde{m})$ is the integration contour depicted in Fig. 2, $\tilde{m} \in \mathbb{R}^+$ being some arbitrary constant, and, similar to before, the integration vanishes due to CAUCHY's theorem [10, 11] and the fact that the integrand $p(z) = \mathrm{e}^{-az^2}$ is an analytic function in the region enclosed by $\tilde{\Gamma}(\tilde{m})$ and on its boundary. We split the integration contour into the four parts $\tilde{\gamma}_1(\tilde{m})$, $\tilde{\gamma}_2(\tilde{m})$, $\tilde{\gamma}_3(\tilde{m})$, and $\tilde{\gamma}_4(\tilde{m})$, so that

$$\begin{aligned}\oint_{\tilde{\Gamma}(\tilde{m})} \mathrm{d}z\, \mathrm{e}^{-az^2} &= \underbrace{\int_{\tilde{\gamma}_1(\tilde{m})} \mathrm{d}z\, \mathrm{e}^{-az^2}}_{\tilde{\gamma}_1(\tilde{m}):\, z=x,\, x\in[-\tilde{m},\tilde{m}]} + \underbrace{\int_{\tilde{\gamma}_2(\tilde{m})} \mathrm{d}z\, \mathrm{e}^{-az^2}}_{\tilde{\gamma}_2(\tilde{m}):\, z=\tilde{m}+\mathrm{i}y,\, y\in[0,-\mathrm{Im}(\frac{b}{2a})]} + \underbrace{\int_{\tilde{\gamma}_3(\tilde{m})} \mathrm{d}z\, \mathrm{e}^{-az^2}}_{\tilde{\gamma}_3(\tilde{m}):\, z=x-\mathrm{i}\mathrm{Im}(\frac{b}{2a}),\, x\in[\tilde{m},-\tilde{m}]} \\ &\quad + \underbrace{\int_{\tilde{\gamma}_4(\tilde{m})} \mathrm{d}z\, \mathrm{e}^{-az^2}}_{\tilde{\gamma}_4(\tilde{m}):\, z=-\tilde{m}+\mathrm{i}y,\, y\in[-\mathrm{Im}(\frac{b}{2a}),0]} \\ &= \int_{-\tilde{m}}^{\tilde{m}} \mathrm{d}x\, \mathrm{e}^{-ax^2} + \int_0^{-\mathrm{Im}(\frac{b}{2a})} \mathrm{d}y\, \mathrm{i}\mathrm{e}^{-a(\tilde{m}+\mathrm{i}y)^2} + \int_{\tilde{m}}^{-\tilde{m}} \mathrm{d}x\, \mathrm{e}^{-a(x-\mathrm{i}\mathrm{Im}(b/(2a)))^2} \\ &\quad + \int_{-\mathrm{Im}(\frac{b}{2a})}^{0} \mathrm{d}y\, \mathrm{i}\mathrm{e}^{-a(-\tilde{m}+\mathrm{i}y)^2}.\end{aligned}$$

Similar to the proof of Lemma 4, we find that the contributions from $\tilde{\gamma}_2(\tilde{m})$ and $\tilde{\gamma}_4(\tilde{m})$ vanish in the limit $\tilde{m} \to \infty$. In particular, we have

$$\begin{aligned}\left| \int_0^{-\mathrm{Im}(\frac{b}{2a})} \mathrm{d}y\, \mathrm{i}\mathrm{e}^{-a(\tilde{m}+\mathrm{i}y)^2} \right| &\leq \int_0^{-\mathrm{Im}(\frac{b}{2a})} \mathrm{d}y\, \left| \mathrm{e}^{-(\mathrm{Re}(a)+\mathrm{i}\mathrm{Im}(a))(\tilde{m}^2-y^2+2\mathrm{i}\tilde{m}y)} \right| \\ &= \mathrm{e}^{-\mathrm{Re}(a)\tilde{m}^2} \int_0^{-\mathrm{Im}(\frac{b}{2a})} \mathrm{d}y\, \mathrm{e}^{\mathrm{Re}(a)y^2+2\mathrm{Im}(a)\tilde{m}y} \underset{\mathrm{Re}(a)>0}{\overset{\tilde{m}\to\infty}{\longrightarrow}} 0,\end{aligned}$$

where we used that $q(y) = \mathrm{e}^{\mathrm{Re}(a)y^2+2\mathrm{Im}(a)\tilde{m}y}$ is continuous (thus bounded) on the compact interval $J = [0, -\mathrm{Im}(b/(2a))]$ for finite (but large) $\tilde{m}$ and that the function $u(\tilde{m}) = \mathrm{e}^{-\mathrm{Re}(a)\tilde{m}^2}$ decreases way faster than $v(\tilde{m}) = \mathrm{e}^{2\mathrm{Im}(a)\tilde{m}y}$ potentially grows in the limit $\tilde{m} \to \infty$ for $y \in J$ and arbitrary $\mathrm{Im}(a)$.[5] Analogously, we find

$$\left| \int_{-\mathrm{Im}(\frac{b}{2a})}^{0} \mathrm{d}y\, \mathrm{i}\mathrm{e}^{-a(-\tilde{m}+\mathrm{i}y)^2} \right| \overset{\tilde{m}\to\infty}{\longrightarrow} 0.$$

Then, using the result of CAUCHY's theorem from above, we find

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} = -\int_{\infty}^{-\infty} \mathrm{d}x\, \mathrm{e}^{-a(x-\mathrm{i}\mathrm{Im}(b/(2a)))^2} \overset{x\equiv\check{x}:=x-\mathrm{i}\mathrm{Im}(\frac{b}{2a})}{=} \int_{-\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})}^{\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})} \mathrm{d}x\, \mathrm{e}^{-ax^2}$$

in the limit $\tilde{m} \to \infty$ and hence with Corollary 3

$$\int_{-\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})}^{\infty-\mathrm{i}\mathrm{Im}(\frac{b}{2a})} \mathrm{d}x\, \mathrm{e}^{-ax^2} = \int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} \overset{\mathrm{Re}(a)>0}{=} \sqrt{\frac{\pi}{a}}. \qquad \square$$

[5] One can for example also check the vanishing of the above expression in the limit $\tilde{m} \to \infty$ with the help of *Mathematica*.

2 Summary

In this article, we provided a semi-rigorous proof for the value of the Gaussian integral,

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-ax^2+bx} = \mathrm{e}^{b^2/(4a)} \sqrt{\frac{\pi}{a}}, \qquad a, b \in \mathbb{C}, \quad \mathrm{Re}(a) \geq 0,$$
$$\mathrm{Re}(a) = 0 \implies \mathrm{Im}(a) \neq 0 \wedge \mathrm{Re}(b) = 0.$$

We proceeded by proving several minor results, first restricting to $b = 0$ and the domain of a to the reals and then extending the validity of this formula to complex values of a by analytic continuation. The final result was then proved by including a linear offset, $b \neq 0$.

Acknowledgements

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