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GREEN's Functions and FOURIER Transforms

A Physicist's Overview

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Abstract

We give a brief overview of GREEN's functions. That is we explain what these functions are, what they are good for and how one can potentially find them – finding a GREEN's function is not limited to this method and in general depends on each special case. The method we will show and clarify with an example is convenient in physics and applicable to many of the problems we deal with in physics. In doing so, we also introduce the FOURIER transform. The word function in GREEN's functions is to be taken with due care. Strictly speaking, a GREEN's function has to be viewed as a distribution, but we will stick to the loose and common terminology here. Furthermore, we will not go into much detail about the mathematics behind the topic and instead focus on giving an introduction to the topic that is supposed to help understanding the concept of GREEN's functions.

For people who want to learn more on this topic, we refer to the books [1], [2] and [3], as well as the internet resources [4] and [5]. This recap is basically based on the given literature and things learned here and there. Another resource that proved useful in preparing this recap was [6].

1 GREEN's Functions and FOURIER Transforms

1.1 Definition of GREEN's Functions

Let us assume we have some linear differential operator $\mathcal{L}_{\mathbf{x}}$ in $\mathbf{x}$ acting on scalar functions, where linear means that for two objects $f(\mathbf{x}, \mathbf{y}, \ldots)$ and $g(\mathbf{x}, \mathbf{y}, \ldots)$ from the space that the operator acts on, we have

$$\mathcal{L}_{\mathbf{x}}[f + g] = \mathcal{L}_{\mathbf{x}}[f] + \mathcal{L}_{\mathbf{x}}[g]. \tag{1.1}$$

The variables $\mathbf{x}, \mathbf{y}, \ldots$ can and in general will be multi-dimensional, denoted by a bold symbol in this section. If we know 'the function' $G(\mathbf{x}, \mathbf{t})$ – actually, there exist operators that have several such functions, while some do not admit any such function – which fulfills the distributional equation

$$\mathcal{L}_{\mathbf{x}}[G(\mathbf{x}, \mathbf{t})] = \delta(\mathbf{x} - \mathbf{t}), \tag{1.2}$$

we can construct a solution $f(\mathbf{x})$ of the equation $\mathcal{L}_{\mathbf{x}}[f(\mathbf{x})] = g(\mathbf{x})$ with appropriate boundary conditions at $\mathbf{x} = \mathbf{a}$ and $\mathbf{x} = \mathbf{b}$. The delta distribution is obviously to be understood as the multi-dimensional delta distribution $\delta^{(n)}$, with n the dimension of the variables $\mathbf{x}$ and $\mathbf{t}$. The function $G(\mathbf{x}, \mathbf{t})$ fulfilling Eq. (1.2) is called the (two-point) GREEN's function corresponding to the operator $\mathcal{L}_{\mathbf{x}}$. The term two-point is supposed to distinguish it from GREEN's functions that can take more than two arguments, which are then referred to as multi-point or n-point GREEN's functions – with n the number of arguments the function depends on. At the latest when taking courses on quantum field theory, this point will get clearer.

1.2 Application of GREEN's Functions

As stated in the previous section, the knowledge of the GREEN's function of an operator gives a method to construct a solution $f(\mathbf{x})$ to the equation

$$\mathcal{L}_{\mathbf{x}}[f(\mathbf{x})] = g(\mathbf{x}). \tag{1.3}$$

More precisely, we claim that

$$f(\mathbf{x}) = \int \mathrm{d}\mathbf{t}\, G(\mathbf{x}, \mathbf{t}) g(\mathbf{t}) \tag{1.4}$$

gives such a solution when the function $G(\mathbf{x}, \mathbf{t})$ fulfills Eq. (1.2). The boundaries of the integral have to be fixed from the boundary conditions of the problem. Here, the bold symbol for the integration variable again suggests that the integral is multi-dimensional if $\mathbf{t}$ is multi-dimensional. And indeed, using the linearity of the operator in the variable $\mathbf{x}$ and upon inserting the ansatz Eq. (1.4), one finds

$$\begin{aligned}\mathcal{L}_{\mathbf{x}}[f(\mathbf{x})] = \mathcal{L}_{\mathbf{x}}\left[\int \mathrm{d}\mathbf{t}\, G(\mathbf{x}, \mathbf{t}) g(\mathbf{t})\right] = \int \mathrm{d}\mathbf{t}\, \mathcal{L}_{\mathbf{x}}[G(\mathbf{x}, \mathbf{t})] g(\mathbf{t}) = \int \mathrm{d}\mathbf{t}\, \delta(\mathbf{x} - \mathbf{t}) g(\mathbf{t}) \\ = g(\mathbf{x}).\end{aligned} \tag{1.5}$$

Note that if the operator $\mathcal{L}_{\mathbf{x}}$ is translation invariant, the GREEN's function takes the form $G(\mathbf{x}, \mathbf{t}) = G(\mathbf{x} - \mathbf{t})$. A linear differential operator is translation invariant if its coefficients do not depend on the variable x. For $\mathbf{x} = x$ one-dimensional, this is for example fulfilled by $\mathcal{L}_{1x} = \partial/\partial x$ because

$$\mathcal{L}_{1(x-x_0)} = \frac{\partial}{\partial(x - x_0)} = \frac{\partial}{\partial x}\frac{\partial x}{\partial(x - x_0)} = \mathcal{L}_{1x}, \tag{1.6}$$

while it is not fulfilled by $\mathcal{L}_{2x} = x\partial/\partial x$ because $\mathcal{L}_{2(x-x_0)} \neq \mathcal{L}_{2x}$. Translational invariance is an important physical property that is demanded in any physical theory that **we** are aware of. According to NOETHER's theorem it corresponds to momentum conservation, which is a fundamental natural property.

1.3 Definition of the Fourier Transform

As of now, we refrain from using bold symbols to indicate multi-dimensional variables and instead explicitly state the dimension. Any variable will be represented by a regular font symbol and the dimension will be clear from context.

The Fourier transform of an integrable function f can be defined for different function spaces $\mathcal{V} \ni f$. A usual choice for the function space $\mathcal{V}$ are the so-called L^p spaces (with a sloppy notation **and** especially not the general case) given by

$$\mathcal{L}^p(\mathbb{R}^n;\mathbb{R}) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}^3} \mathrm{d}^n x \, |f(x)|^p < \infty \right\}. \tag{1.7}$$

The Fourier transform is most easily defined for $f \in \mathcal{L}^1(\mathbb{R}^n;\mathbb{R})$, according to

$$\mathcal{F}[f](y) = \hat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{d}^n x \, f(x) \, \mathrm{e}^{-\mathrm{i}x\cdot y}, \tag{1.8}$$

with the canonical scalar product $x \cdot y = \sum_i^n x_i y_i$ on $\mathbb{R}^n$. This obviously gives a function for which $y \in \mathbb{R}^n$. The inverse Fourier transform is then given by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{d}^n y \, \mathcal{F}[f](y) \, \mathrm{e}^{\mathrm{i}y\cdot x} \tag{1.9}$$

Note that applying the transformation twice yields $\mathcal{F}[\mathcal{F}[f]](x) = f(-x)$ and is thus different from the inverse transform. The factor of $1/(2\pi)^{n/2}$ in front of the integral differs for various conventions. In our case, it warrants that the transform and inverse transform are symmetrized in the sense that both carry the same factor. Other definitions of the Fourier transform might even carry the factors of 2π in the exponential function in the integral instead. Denoting the partial derivative with respect to the i-th variable – that is $\partial/\partial x_i$ – as ∂_i, it is easy to show that

$$\mathcal{F}[\partial_i f](y) = \mathrm{i} y_i \mathcal{F}[f](y). \tag{1.10}$$

The Fourier transform of the n-dimensional delta distribution is easily calculated to be

$$\mathcal{F}[\delta^{(n)}](y) = \frac{1}{(2\pi)^{n/2}}. \tag{1.11}$$

Consequently, the delta distribution can be expressed via the inverse Fourier transform of $1/(2\pi)^{n/2}$, that is

$$\delta^{(n)}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{d}^n t \, \mathrm{e}^{\mathrm{i}t\cdot x}. \tag{1.12}$$

2 The Laplace and d'Alembert Operator

2.1 The Laplace Operator

As an example, let us consider the Laplace operator in three dimensions

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\,, \tag{2.1}$$

and the problem of determining the potential $\Phi(\vec{x})$ that fulfills the Poisson equation

$$\Delta\Phi(\vec{x}) = \frac{\rho(\vec{x})}{\epsilon_0} \tag{2.2}$$

for some charge density $\rho(\vec{x})$. In Eq. (1.4), we reduced the problem of finding $\Phi(\vec{x})$ to the problem of solving the integral

$$\Phi(\vec{x}) = \int \mathrm{d}^3x'\, G_\Delta(\vec{x},\vec{x}')\rho(\vec{x}'), \tag{2.3}$$

where $G_\Delta(\vec{x},\vec{x}')$ is the Green's function of the Laplace operator, $\Delta G_\Delta(\vec{x},\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}')$. First note that the Laplace operator is translation invariant in the sense explained before, which means that our Green's function is of the form $G_\Delta(\vec{x},\vec{x}') = G_\Delta(\vec{x}-\vec{x}') = G_\Delta(\vec{z})$ for $\vec{z} = \vec{x}-\vec{x}'$. In order to obtain the Green's function of the Laplace operator, we start with the equation defining said function, that is

$$\Delta G_\Delta(\vec{z}) = \delta^{(3)}(\vec{z}), \tag{2.4}$$

and Fourier transform both sides of the equation to find

$$\mathcal{F}[\Delta G_\Delta](\vec{y}) = \mathcal{F}[\delta^{(3)}](\vec{y}). \tag{2.5}$$

Using Eq. (1.10) and Eq. (1.11), we find

$$-(y_1^2+y_2^2+y_3^2)\mathcal{F}[G_\Delta](\vec{y}) = \frac{1}{(2\pi)^{3/2}}, \tag{2.6}$$

such that with the inverse FOURIER transform we have

$$G_\Delta(\vec{z}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{d}^3y\,\mathcal{F}[G_\Delta](\vec{y})\mathrm{e}^{\mathrm{i}\vec{y}\cdot\vec{z}} = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}^3y\,\frac{1}{|\vec{y}|^2}\mathrm{e}^{\mathrm{i}\vec{y}\cdot\vec{z}}. \tag{2.7}$$

Using spherical coordinates for $\vec{y}=(y_1,y_2,y_3)^\intercal$, making use of $\vec{x}\cdot\vec{y}=xy\cos\theta$ with θ the angle between $\vec{x}$ and $\vec{y}$, $x=|\vec{x}|$, $y=|\vec{y}|$, and keeping in mind that we get a JACOBIan determinant from the transformation of variables, we find

$$\begin{aligned} G_\Delta(\vec{z}) &= -\frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}y \int_{-1}^{1}\mathrm{d}\cos\theta \int_0^{2\pi}\mathrm{d}\phi\,\frac{\mathrm{e}^{\mathrm{i}yz\cos\theta}}{y^2}y^2 \\ &= -\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}y\,\frac{1}{\mathrm{i}yz}\left(\mathrm{e}^{\mathrm{i}yz}-\mathrm{e}^{-\mathrm{i}yz}\right) = -\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}y\,\frac{2\sin(yz)}{yz} \\ &= -\frac{1}{2\pi^2}\frac{\pi}{2z} = -\frac{1}{4\pi z}. \end{aligned} \tag{2.8}$$

The value of the familiar integral $\int \sin x/x$ is assumed to be known here, while in fact it is not easy to solve at all. In particular, we thus found that the GREEN's function is only a function of the absolute value of its argument, $G_\Delta(\vec{z})=G_\Delta(z)=-1/(4\pi z)$.

Note that the same result can (of course) be obtained when instead of Eq. (2.5), we rewrite Eq. (2.4) by using Eq. (1.9) and Eq. (1.12), apply the LAPLACE operator to the integral and read of $\mathcal{F}[G_\Delta](\vec{y})$.

We now want to use the GREEN's function we just obtained to solve Eq. (2.3). To this end, we write as before $G_\Delta(z)=G_\Delta(|\vec{x}-\vec{x}'|)=G_\Delta(\vec{x},\vec{x}')$ and find that the solution to the POISSON equation is given by

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\int \mathrm{d}^3x'\,\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{2.9}$$

For a given charge density $\rho(\vec{x})$, the remaining task is now to solve this integral.

2.2 The D'ALEMBERT Operator

Just to resolve any possible confusion, we briefly discuss the D'ALEMBERT Operator which we also discussed in class. It is given by

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta, \tag{2.10}$$

and by the same argument translation invariant. Hence, we know that its GREEN's function takes the form $G_\Box(\underline{x}, \underline{x}') = G_\Box(\underline{x} - \underline{x}') = G_\Box(\underline{z})$, with $\underline{z} = \underline{x} - \underline{x}'$ all four-dimensional (time and three spatial components) vectors. It furthermore fulfills its defining equation

$$G_\Box(\underline{z}) = \delta^{(4)}(\underline{z}) = \delta^{(3)}(\vec{z})\delta(z^0), \tag{2.11}$$

where $\vec{z}$ are the spatial components of the four-vector $\underline{z}$ and z^0 is its time-component. In class, we wrote this as

$$G_\Box(\vec{r}, t) = \delta^{(3)}(\vec{r})\delta(t), \tag{2.12}$$

which is completely equivalent. The variable t (time) is not be confused with the $\mathbf{t}$ in Eq. (1.2). In the case of the D'ALEMBERT operator, the corresponding integral we solved in Eq. (2.8) was the tough part – that is where complex analysis came into play. The principle behind finding a GREEN's function is way easier to understand and the example of the D'ALEMBERT operator is just a poor choice to introduce the topic in my opinion. What we found is

$$G_\Box(r, t) = \theta(t)\frac{1}{4\pi r}\left[\delta(t - \frac{r}{c}) - \delta(t + \frac{r}{c})\right] = \frac{1}{4\pi r}\delta(t - \frac{r}{c}), \tag{2.13}$$

thus in particular again that the function only depends on the absolute values $r = |\vec{r}|$ and t.

3 Summary

We defined Green's function in a manner sufficient for physicists and motivated their introduction. That is we showed how they can be used to solve differential equations like the Poisson equation. Defining Fourier transforms, we then also gave an example of how to obtain the Green's function corresponding to the Laplace operator. Using this Green's function, we gave a solution of the Poisson equation. Finally, we briefly discussed the d'Alembert operator.

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Bibliography

[1] G. B. Arfken et al.: Mathematical Methods for Physicists – *A Comprehensive Guide.*

[2] T. Rother: Green's Functions in Classical Physics – *Lecture Notes in Physics.*

[3] D. G. Duffy: Green's Functions with Application – *Studies in Advanced Mathematics.*

[4] Stover, Christopher: Green's Function – *From MathWorld – A Wolfram Web Resource, created by Eric W. Weisstein.*
http://mathworld.wolfram.com/GreensFunction.html

[5] Green's Function – *Wikipedia.*
https://en.wikipedia.org/wiki/Green%27s_function

[6] Fourier-Transformation – *Wikipedia.*
https://de.wikipedia.org/wiki/Fourier-Transformation