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Rotating Charged Sphere

Problem with Suggested Solution

Marvin Zanke

www.Physics-and-Stuff.com

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«Il n'y a pas plus sourd que celui qui ne veut pas entendre.»

– *A french saying*

1 Problem: A Rotating Charged Sphere

The following exercise can be found in a similar form in [2, p. 247].

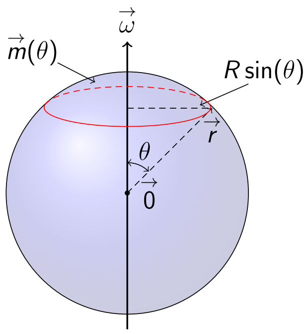

Figure 1: The rotating sphere with a circular segment exemplarily marked in red. Throughout the whole problem, we will work in spherical coordinates. TikZ template taken from [1] and modified to make it better fit this problem.

A uniformly charged sphere with radius R and surface charge density σ is rotating in the vacuum with constant angular velocity $\omega = |\vec{\omega}|$ around the symmetry axis z, i.e. $\vec{\omega} = \omega\hat{e}_z$. We choose the center of the sphere at the origin $(0, 0, 0)$ (see Fig. 1).

a) Calculate the magnetic dipole moment $\vec{m}$ of the sphere. You can can break the sphere up in thin slices perpendicular to the rotation axis (see Fig. 1). Then you can use the result for the magnetic moment of a circular conducting ring and integrate the slices over the whole sphere.

b) Determine the vector potential $\vec{A}(\vec{r})$. To this end, expand the vector potential in spherical harmonics and write $\sin(\theta) \cdot \hat{e}_\phi$ in terms of spherical harmonics. Having done so, you have to use their orthogonality. *Hint: Make a distinction between the two cases of inside and outiside the sphere.*

c) Calculate the magnetic field $\vec{B}(\vec{r})$ in the inside and for the outside of the sphere. Show that outside of the sphere, the magnetic field takes the form of a magnetic dipole. Determine the corresponding magnetic dipole moment $\vec{m}$ and compare to part a).

We now want to replace the sphere by a uniformly charged ball. The radius remains R and the charge on the ball is Q. The ball is, as before, rotating around the z-axis with the same angular velocity. That is, Fig. 1 basically stays the same.

d) Determine the vector potential $\vec{A}(\vec{r})$ for the case of said rotating ball. You can use the result obtained in b) and replace $R \to r'$, $\sigma \to \rho \mathrm{d}r'$, with ρ the charge density of the ball. If you then integrate over r', you will get the desired result. Explain why! *Hint: Take care when using the result of b). For the inner of the ball, two different contributions have to be taken in account.*

e) Obtain the magnetic field $\vec{B}(\vec{r})$ inside and outside of the ball. Analogously to before, write the field outside of the sphere as a magnetic dipole field and determine the corresponding magnetic dipole moment $\vec{m}$.

2 Suggested Solution

Throughout the whole problem, we will work in spherical coordinates,

$$\vec{r}(r,\theta,\phi) = \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix}, \tag{1}$$

where we will omit the arguments of $\vec{r}$. We follow the usual convention that $r = |\vec{r}|$.

We start by collecting the formulae we will need to solve this exercise. They can be found in [2, 3] or any other textbook on classical electrodynamics. The magnetic moment $\vec{m}$ of a circular conducting ring in the x-y-plane with radius R is given by

$$\vec{m} = I\pi R^2 \hat{e}_z, \tag{2}$$

where I is the current flowing through the ring.

In the static case (no time-dependence in the current density), the vector potential $\vec{A}(\vec{r})$ of a configuration with current density $\vec{j}(\vec{x})$ can be calculated using the formula

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3 r' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}. \tag{3}$$

The current density of a charge density ρ moving with velocity $\vec{v}$ is given by

$$\vec{j}(\vec{r}) = \rho\vec{v}. \tag{4}$$

In the context of potentials and GREEN's functions, one can derive (also derived in one of our other exercises)

$$\int_{\mathbb{R}^3} \mathrm{d}^3r\,' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \int_0^{\infty} dr'\, r'^2 \frac{r_<^l}{r_>^{l+1}} \int_{S^2} \mathrm{d}\Omega' Y_{lm}^*(\theta',\phi')\rho(\vec{r}\,'), \tag{5}$$

with $r_< = \min(|\vec{r}|,|\vec{r}\,'|)$, $r_> = \max(|\vec{r}|,|\vec{r}\,'|)$, the spherical harmonics $Y_{lm}(\theta,\phi)$ and the 2-sphere S^2.

Given a vector potential $\vec{A}(\vec{r})$, the magnetic field $\vec{B}(\vec{r})$ is given by the curl of the vector potential, that is

$$\vec{B}(\vec{r}) = \nabla \times \vec{A}(\vec{r}). \tag{6}$$

While in cartesian coordinates, the curl takes a conceivably simple form, in spherical coordinates it is given by

$$\begin{aligned}\nabla\times\vec{A}(\vec{r}) &= \hat{e}_r \frac{1}{r\sin(\theta)} \left[\frac{\partial}{\partial\theta}\left(\sin(\theta)A_\phi(\vec{r})\right) - \frac{\partial A_\theta(\vec{r})}{\partial\phi}\right] \\ &+ \hat{e}_\theta \left[\frac{1}{r\sin(\theta)}\frac{\partial A_r(\vec{r})}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi(\vec{r})\right)\right] + \hat{e}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta(\vec{r})\right) - \frac{\partial A_r(\vec{r})}{\partial\theta}\right].\end{aligned} \tag{7}$$

Note how the derivatives do not act on the basis vectors. The magnetic field generated by a magnetic dipole moment $\vec{m}$ is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{e}_r(\vec{m}\cdot\hat{e}_r) - \vec{m}}{r^3}. \tag{8}$$

2.1 a) Magnetic Dipole Moment

As explained in the problem, we break the sphere up in circular rings and calculate their magnetic moment to later integrate over the angle θ and find the magnetic moment of the full sphere. Using Eq. (2), we find for the magnetic moment $\vec{m}(\theta)$ of the ring in θ-direction

$$\vec{m}(\theta) = r^2(\theta)\pi I(\theta)\hat{e}_z. \tag{9}$$

The radius of the ring and the current obviously depend on the angle (see Fig. 1). The current emerges from the charge on the sphere rotating around the z-axis. The angular velocity is connected to the time T necessary for one revolution (period) via $\omega = \frac{2\pi}{T}$. One can then deduce

$$I(\theta) = \frac{Q(\theta)}{T} = \sigma A(\theta)\frac{\omega}{2\pi}, \tag{10}$$

where $A(\theta) = 2\pi R^2 \sin(\theta)$ is the surface area of the ring in θ-direction. This can be deduced when considering the integral over the sphere with radius R

$$A = \int_{S_R} \mathrm{d}A = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \, \underbrace{R^2 \sin(\theta)}_{\text{Jacobian}} = 2\pi \int_0^\pi \mathrm{d}\theta \, R^2 \sin(\theta), \tag{11}$$

such that $\mathrm{d}A_{\text{Ring}}(\theta) = 2\pi R^2 \sin(\theta)\mathrm{d}\theta$ and thus $A(\theta) = 2\pi R^2 \sin(\theta)$ for a fixed angle θ as claimed above. Inserting $r(\theta) = R\sin(\theta)$ and Eq. (10) into Eq. (9), we find

$$\vec{m}(\theta) = R^2 \sin^2(\theta)\pi\sigma A(\theta)\frac{\vec{\omega}}{2\pi} = \frac{2\pi R^4 \sin^3(\theta)\sigma\vec{\omega}}{2}. \tag{12}$$

Integrating this over θ, we find the magnetic moment of the sphere to be given by

$$\vec{m} = \int_0^\pi \mathrm{d}\theta \, \vec{m}(\theta) = \pi R^4 \sigma \vec{\omega} \underbrace{\int_0^\pi \mathrm{d}\theta \, \sin^3(\theta)}_{4/3} = \underbrace{\frac{4}{3}\pi R^3}_{V_{\text{Ball}}}(R\sigma\vec{\omega}). \tag{13}$$

2.2 b) Vector Potential

To calculate the vector potential, we will use Eq. (3). For the current density of our problem, we use Eq. (4), yielding

$$\vec{j}(\vec{r}) = \sigma\delta(r - R)\vec{v}(\vec{r}) = \sigma\delta(r - R)(\vec{\omega} \times \vec{r}) = \sigma\delta(r - R)R\omega\sin(\theta) \underbrace{\begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix}}_{\hat{e}_\phi} \tag{14}$$

because the charge density becomes a surface charge density and the delta distribution ensures that there only is a contribution on the sphere. Here, we introduced $r = |\vec{r}|$ as mentioned below Eq. (1) and inserted $\vec{r}$ in spherical coordinates, Eq. (1). Inserting this into the vector potential Eq. (3) and using Eq. (5) (replacing $\rho(\vec{r}\,')$ with $\vec{j}(\vec{r}\,')$), we find

$$\begin{aligned} \vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}^3 r' \, \frac{\sigma\delta(r' - R)R\omega\sin(\theta')\hat{e}_{\phi'}}{|\vec{r} - \vec{r}\,'|} \\ &= \frac{\mu_0 \sigma R^3 \omega}{4\pi} \frac{r_<^l}{r_>{}^{l+1}} \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) \underbrace{\int_{-1}^{1} \mathrm{d}\cos\theta' \int_0^{2\pi} \mathrm{d}\phi' Y_{lm}^*(\theta', \phi') \sin(\theta')\hat{e}_{\phi'}}_{(*)} \end{aligned} \tag{15}$$

where because of the delta distribution, the r'-integration evaluates the integrand at $r' = R$ and we have $r_< = \min(|\vec{r}|, R)$, $r_> = \max(|\vec{r}|, R)$. We now rewrite $\sin(\theta')\hat{e}_{\phi'}$ in terms of spherical harmonics, according to

$$\begin{aligned} \sin(\theta')\hat{e}_{\phi'} &= \sin(\theta') \begin{pmatrix} -\sin(\phi') \\ \cos(\phi') \\ 0 \end{pmatrix} = \sin(\theta') \begin{pmatrix} \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\phi'} - \mathrm{e}^{-\mathrm{i}\phi'}) \\ \frac{1}{2}(\mathrm{e}^{\mathrm{i}\phi'} + \mathrm{e}^{-\mathrm{i}\phi'}) \\ 0 \end{pmatrix} \\ &= \sqrt{\frac{8\pi}{3}} \begin{pmatrix} \frac{1}{2\mathrm{i}}\left[Y_{11}(\theta', \phi') + Y_{1-1}(\theta', \phi')\right] \\ -\frac{1}{2}\left[Y_{11}(\theta', \phi') - Y_{1-1}(\theta', \phi')\right] \\ 0 \end{pmatrix}, \end{aligned} \tag{16}$$

such that the $(*)$ equation from above becomes

$$\begin{aligned} &\int_{-1}^{1} \mathrm{d}\cos\theta' \int_{0}^{2\pi} \mathrm{d}\phi'\, Y^*_{lm}(\theta', \phi') \sin(\theta')\hat{e}_{\phi'} \\ &= \sqrt{\frac{8\pi}{3}} \int_{-1}^{1} \mathrm{d}\cos\theta' \int_{0}^{2\pi} \mathrm{d}\phi'\, Y^*_{lm}(\theta', \phi') \begin{pmatrix} \frac{1}{2\mathrm{i}}\left[Y_{11}(\theta', \phi') + Y_{1-1}(\theta', \phi')\right] \\ -\frac{1}{2}\left[Y_{11}(\theta', \phi') - Y_{1-1}(\theta', \phi')\right] \\ 0 \end{pmatrix} \\ &= \sqrt{\frac{8\pi}{3}} \delta_{l1} \begin{pmatrix} \frac{1}{2\mathrm{i}}\left(\delta_{m1} + \delta_{m-1}\right) \\ -\frac{1}{2}\left(\delta_{m1} - \delta_{m-1}\right) \\ 0 \end{pmatrix}. \end{aligned} \tag{17}$$

Inserting this back into Eq. (15), we find the vector potential to be given by

$$\begin{aligned} \vec{A}(\vec{r}) &= \frac{\mu_0 \sigma R^3 \omega}{3} \frac{r_<}{{r_>}^2} \underbrace{\sqrt{\frac{8\pi}{3}} \begin{pmatrix} \frac{1}{2\mathrm{i}}\left[Y_{11}(\theta, \phi) + Y_{1-1}(\theta, \phi)\right] \\ -\frac{1}{2}\left[Y_{11}(\theta, \phi) - Y_{1-1}(\theta, \phi)\right] \\ 0 \end{pmatrix}}_{\sin(\theta)\hat{e}_\phi \text{ (see Eq. (16))}} \\ &= \begin{cases} \frac{\mu_0\sigma\omega}{3} r R \sin(\theta)\hat{e}_\phi & \text{for } r \leq R, \\ \frac{\mu_0\sigma\omega}{3} \frac{R^4}{r^2} \sin(\theta)\hat{e}_\phi & \text{for } r \geq R, \end{cases} \end{aligned} \tag{18}$$

for the two cases inside and outside of the sphere.

2.3 c) Magnetic Field

For the magnetic field, we now use the solutions we found for the vector potential and insert them into Eq. (6) under the use of Eq. (7). The solution inside of the sphere was found to be $\vec{A}_{\text{in}}(\vec{r}) = \frac{\mu_0 \sigma \omega}{3} r R \sin(\theta) \hat{e}_\phi$ – which only has a ϕ-component – such that we find

$$\begin{aligned}\vec{B}_{\text{in}}(\vec{r}) &= \hat{e}_r \frac{1}{r\sin(\theta)} \partial_\theta \left(\frac{\mu_0 \sigma \omega}{3} r R \sin^2(\theta) \right) - \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu_0 \sigma \omega}{3} r^2 R \sin(\theta) \right) \\ &= \frac{2}{3} \mu_0 \sigma \omega R \underbrace{\left(\cos(\theta) \hat{e}_r - \sin(\theta) \hat{e}_\theta \right)}_{\hat{e}_z} . \end{aligned} \tag{19}$$

For the solution outside of the sphere, we found $\vec{A}_{\text{out}}(\vec{r}) = \frac{\mu_0 \sigma \omega}{3} \frac{R^4}{r^2} \sin(\theta) \hat{e}_\phi$. The magnetic field is then found to be

$$\begin{aligned}\vec{B}_{\text{out}}(\vec{r}) &= \hat{e}_r \frac{1}{r\sin(\theta)} \partial_\theta \left(\frac{\mu_0 \sigma \omega}{3} \frac{R^4}{r^2} \sin^2(\theta) \right) - \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu_0 \sigma \omega}{3} \frac{R^4}{r} \sin(\theta) \right) \\ &= \frac{\mu_0 \sigma \omega}{3} \frac{R^4}{r^3} \left(2\cos(\theta) \hat{e}_r + \sin(\theta) \hat{e}_\theta \right) . \end{aligned} \tag{20}$$

To find the magnetic dipole moment and bring this magnetic field into the form Eq. (8), we note that

$$\hat{e}_r (\hat{e}_r \cdot \vec{\omega}) = \hat{e}_r \left(\omega \cos(\theta) \right) \iff \cos(\theta) \hat{e}_r = \frac{\hat{e}_r \cdot \vec{\omega}}{\omega} \hat{e}_r \tag{21}$$

as well as

$$\begin{aligned}\sin(\theta)\hat{e}_\theta = \sin(\theta) \begin{pmatrix} \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \\ -\sin(\theta) \end{pmatrix} &= \cos(\theta) \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ = \cos(\theta) \hat{e}_r - \hat{e}_z . \end{aligned} \tag{22}$$

Inserting this back into Eq. (28), we find

$$\vec{B}_{\text{out}}(\vec{r}) = \frac{\mu_0 \sigma \omega}{3} \frac{R^4}{r^3} \left(3 \frac{\hat{e}_r \cdot \vec{\omega}}{\omega} \hat{e}_r - \hat{e}_z \right) = \frac{\mu_0 \sigma}{3} R^4 \frac{3\left(\hat{e}_r \cdot \vec{\omega} \right) \hat{e}_r - \vec{\omega}}{r^3} \tag{23}$$

and a comparison with Eq. (8) gives – in accordance with Eq. (13) from a) – that

$$\vec{m} = \frac{4\pi}{3} R^3 \left(R \sigma \vec{\omega} \right) . \tag{24}$$

2.4 d) Vector Potential

Replacing the sphere by a ball, the radius R has to be replaced by r' because the current density does not contain a delta distribution which sets $r' = R$ anymore. Furthermore, the surface charge density σ becomes a charge density ρ and we have to integrate the resulting equations over the radius r', because we can think of the ball as an infinite amount of spheres reaching from the origin to the radius R of the ball. That is why we now replace $R \to r'$, $\sigma \to \rho$ plus an integration $\mathrm{d}r'$.

For the charge density, we obtain $\rho = \frac{Q}{V} = \frac{3Q}{4\pi R^3}$. In order to find the vector potential of the ball (inside and outside), we can use the result for the sphere obtained in b) and integrate it over r'. Care has to be taken for the solution inside of the ball – if you want to find the vector potential at a point $\vec{r}$, that is $\vec{A}(\vec{r})$, bear in mind that when integrating over the spheres with radius $|\vec{r}\,'| = r'$, the vector $\vec{r}$ will lie outside of these spheres as long as the integration variable is smaller than $|\vec{r}| = r$. In this region, the outside solution has to be used, while for the integration region where $r' > r$, the inside solution is the correct choice. For inside of the ball, we thus find

$$\begin{aligned}\vec{A}_{\text{in}}(\vec{r}) &= \frac{\mu_0 \rho \omega}{3} \left[\underbrace{\int_0^r \mathrm{d}r' \, \frac{r'^4}{r^2}}_{r' \leq r} + \underbrace{\int_r^R \mathrm{d}r' \, r r'}_{r' \geq r} \right] \sin(\theta) \hat{e}_\phi \\ &= \frac{\mu_0 Q \omega}{8\pi} \left(\frac{r}{R} - \frac{3}{5} \frac{r^3}{R^3} \right) \sin(\theta) \hat{e}_\phi \end{aligned} \tag{25}$$

after inserting the charge density. For the outside of the ball, we analogously find

$$\begin{aligned}\vec{A}_{\text{out}}(\vec{r}) &= \frac{\mu_0 \rho \omega}{3} \int_0^R \mathrm{d}r' \, \frac{r'^4}{r^2} \sin(\theta) \hat{e}_\phi \\ &= \frac{\mu_0 Q \omega}{20\pi} \frac{R^2}{r^2} \sin(\theta) \hat{e}_\phi . \end{aligned} \tag{26}$$

2.5 e) Magnetic Field

For the magnetic field, we proceed as in c), that is use Eq. (6) with Eq. (7). The solution inside of the ball was found to be $\vec{A}_{\text{in}}(\vec{r}) = \frac{\mu_0 Q \omega}{8\pi} \left(\frac{r}{R} - \frac{3}{5} \frac{r^3}{R^3} \right) \sin(\theta) \hat{e}_\phi$ and we find

$$\begin{aligned}\vec{B}_{\text{in}}(\vec{r}) &= \hat{e}_r \frac{1}{r\sin(\theta)} \frac{\mu_0 Q\omega}{8\pi} \left(\frac{r}{R} - \frac{3}{5}\frac{r^3}{R^3} \right) \partial_\theta \left(\sin^2(\theta) \right) \\ &\quad - \hat{e}_\theta \frac{1}{r} \sin(\theta) \frac{\mu_0 Q\omega}{8\pi} \frac{\partial}{\partial r} \left(\frac{r^2}{R} - \frac{3}{5}\frac{r^4}{R^3} \right) \\ &= \frac{\mu_0 Q\omega}{4\pi} \left(\frac{1}{R} - \frac{3}{5}\frac{r^2}{R^3} \right) \underbrace{\left(\cos(\theta)\hat{e}_r - \sin(\theta)\hat{e}_\theta \right)}_{\hat{e}_z} + \frac{\mu_0 Q\omega}{4\pi} \frac{3}{5} \frac{r^2}{R^3} \sin(\theta)\hat{e}_\theta. \end{aligned} \tag{27}$$

For the solution outside of the sphere, we found $\vec{A}_{\text{out}}(\vec{r}) = \frac{\mu_0 Q\omega}{20\pi}\frac{R^2}{r^2}\sin(\theta)\hat{e}_\phi$. The magnetic field is then found to be

$$\begin{aligned}\vec{B}_{\text{out}}(\vec{r}) &= \hat{e}_r \frac{1}{r\sin(\theta)} \frac{\mu_0 Q\omega}{20\pi} \frac{R^2}{r^2} \partial_\theta \left(\sin^2(\theta) \right) - \hat{e}_\theta \frac{1}{r} \sin(\theta) \frac{\mu_0 Q\omega}{20\pi} \frac{\partial}{\partial r} \left(\frac{R^2}{r} \right) \\ &= \frac{\mu_0 Q\omega}{20\pi} \frac{R^2}{r^3} \left(2\cos(\theta)\hat{e}_r + \sin(\theta)\hat{e}_\theta \right) \\ &= \frac{\mu_0 Q\omega}{20\pi} \frac{R^2}{r^3} \left(3\frac{\hat{e}_r \cdot \vec{\omega}}{\omega}\hat{e}_r - \hat{e}_z \right) \\ &= \frac{\mu_0 Q R^2}{20\pi} \frac{3\left(\hat{e}_r \cdot \vec{\omega}\right)\hat{e}_r - \vec{\omega}}{r^3}, \end{aligned} \tag{28}$$

where we used the same trick as in c) for the last steps. A comparison with Eq. (8) then gives

$$\vec{m} = \frac{1}{5} Q R^2 \vec{\omega} \tag{29}$$

for the magnetic dipole moment. This also could have been calculated analogously to a) under the given replacements (see the accordance we found in Eq. (24)).

Bibliography

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