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Multi-Dimensional Integration

A Physicist's Recap

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Abstract

We give a summary of multi-dimensional integration. That is **line integrals, surface integrals, volume integrals** and everything that is connected to these. We also talk about Gauss' **theorem** and Stokes' **theorem**. The reader is expected to already have some knowledge on integration. Even though we try to be as general as necessary, we leave general cases like integration over abstract vector spaces and other details to the Mathematicians. Instead, we stick to integration over $\mathbb{R}^n$. Because of the isomorphism $\mathbb{R}^2 \cong \mathbb{C}$, portions of this review might be applicable to an integration in the complex plane under (slight) modifications. We will not further talk about this but it shall be noted that Cauchy's integral theorem and the residue theorem play a big role in complex analysis. At times, this summary might lack mathematical rigor for the sake of understanding and intuition.

For people who are interested in a mathematical approach to this topic, I can recommend the books [1], [2] and [3], as well as the scripts [4], [5] and [6]. They cover everything you need to know to kick off your career in Analysis. Unfortunately, as far as I know, they are only available in German. The Wikipedia pages [7], [8], [9] and [10] also do a really good job on these topics and contain some useful graphics and animations. They are available in German and English. This recap is basically based on the given literature and things learned here and there.

1 Introduction

We want to give a recap of multi-dimensional integration to the extent necessary for the daily life of a Physicist. We shall consider only such functions, that allow for an integration (are (Riemann) **integrable**). That is functions, for which it makes sense to define the (Riemann) integral and the integral exists (either in the proper or improper sense). Usually, physicists deal with functions that are **smooth**, i.e. that can be derived any number of times, with the derivatives being continuous. An example of the class of functions we shall not consider is the Dirichlet function

$$\begin{aligned} D: \quad & \mathbb{R} \to \mathbb{R}, \\ & x \mapsto D(x) = \begin{cases} 1, & \text{for } x \in \mathbb{Q} \\ 0, & \text{for } x \in \mathbb{R} \backslash \mathbb{Q}, \end{cases} \end{aligned}$$

which can be integrated using the Lebesgue integral, yielding

$$\int_{\mathbb{R}} \mathrm{d}x \, D(x) = 0$$

because $\mathbb{Q}$ is a countable and thus null set.

1.1 Terminology

To standardize the terminology we use, we give an overview of the different terms and notations that exist for the functions we will talk about.

For some function

$$f: \mathbb{V} \to \mathbb{W},$$

with $\mathbb{V}$ and $\mathbb{W}$ being vector spaces, we refer to $\mathbb{V}$ as the **domain** and to $\mathbb{W}$ as the **codomain**. As mentioned in the abstract, we will restrict ourselves to the cases where $\mathbb{V} = \mathbb{R}^n$ and $\mathbb{W} = \mathbb{R}^m$ for $\{n, m\} \in \mathbb{N}$.

A **scalar(-valued) function** is a function that maps

$$f:\, \mathbb{R} \to \mathbb{R},$$

while a **scalar field** is a function

$$f:\, \mathbb{R}^n \to \mathbb{R},$$

where often $n \leq 4$ for physical applications.

A **vector(-valued) function** is a function that maps

$$f:\, \mathbb{R} \to \mathbb{R}^m,$$

while a **vector field** is a function

$$f:\, \mathbb{R}^n \to \mathbb{R}^m,$$

where again, $n \leq 4$ and $m \leq 4$ for most of the physical applications. So in other words: if we are talking about a field, we refer to the domain of the function being multi-dimensional. The terms scalar(-valued) and vector(-valued) function refer to the dimension of the codomain. If you are not going into the field of String Theory, you will probably be left with at most four-dimensional vector spaces for the rest of your studies. Even though this might be the correct way to refer to these types of functions, you will often just read 'function' for any of them. While sometimes, we might explicitly use the term 'field', this is also the convention we stick to.

There are two more abuses of notation we will commit, where in both cases, it will be clear from context what is meant. Firstly, vectors will sometimes be written with an arrow on top of them and sometimes not. Secondly, integrations over some domain D often need a **parametrization** of that domain. A parametrization is a function that parametrizes the domain using a set of parameters (*who would have thought*). That is, a parametrization of D maps a set of parameters $(s, t, \ldots)$ according to

$$(s, t, \ldots) \mapsto D(s, t, \ldots) = (x_1(s, t, \ldots), x_2(s, t, \ldots), \ldots).$$

At least locally, a line can always be parametrized by one parameter, a surface by two parameters, and so on. Globally, it might be difficult to find one single parametrization. The mentioned abuse is, that we label the function that gives said parametrization with the same letter $D(s, t, \ldots)$ as we label the domain D with.

2 What Types of Functions can be Integrated?

First of all, note that there is always the possibility to integrate a function componentwise. That is, for a function $f_0 : \mathbb{R}^n \to \mathbb{R}^m$, we can integrate each component of the vector $f_0(x_1, \dots, x_n) = \begin{pmatrix} f_0^1(x_1, \dots, x_n) \\ f_0^2(x_1, \dots, x_n) \\ \vdots \\ f_0^m(x_1, \dots, x_n) \end{pmatrix}$ to find $I_0 = \begin{pmatrix} I_0^1 \\ I_0^2 \\ \vdots \\ I_0^m \end{pmatrix}$, which for $i \in \{1, \dots, m\}$ is given by the integrals

$$I_0^i = \int_{a_1}^{b_1} \mathrm{d}x_1 \int_{a_2}^{b_2} \mathrm{d}x_2 \dots \int_{a_n}^{b_n} \mathrm{d}x_n \, f_0^i(x_1, \dots, x_n)$$

over the domains $[a_j, b_j] \subseteq \mathbb{R}$, $j \in \{1, \dots, n\}$. The resulting vector I_0 has the same dimension m as the codomain of $f_0(x_1, \dots x_n)$. This is a rather boring case and we will focus on understanding other cases of possible integrations. More on this, on the notation used, and a simpler notation will follow in the next chapters.

2.1 One-Dimensional Integration

We first consider the well-known integration of functions

$$\begin{aligned} f_1 : \quad & \mathbb{R} \to \mathbb{R}, \\ & x \mapsto f_1(x) \end{aligned}$$

over a domain $[a, b] \subseteq \mathbb{R}$. Note, that the domain the function is defined on could as well be restricted to a subset of $\mathbb{R}$, as long as the integration domain does not exceed the function's domain. We then have

$$I_1 = \int_a^b \mathrm{d}x \, f_1(x).$$

There is a variety of notations you may find here. While Physicists usually write the **measure** $\mathrm{d}x$ right after the integral it belongs to, Mathematicians often write the integral measure after the function that is to be integrated. Another thing you might see is, that the domain is written under the integral as $[a, b]$ instead of the lower and upper boundaries being written below and above the integral sign. That means an equivalent notation would be

$$I_1 = \int_{[a,b]} f_1(x)\, \mathrm{d}x$$

or any mix of these conventions.

We know these **one-dimensional integrations** so well, that it might appear clear to us what is meant, while an integral

$$I_3 = \int_{\mathcal{C}_3} \mathrm{d}s\, f_3(\vec{x})$$

with $f_3 : \mathbb{R}^n \to \mathbb{R}$ over some curve $\mathcal{C}_3$ parametrized by $\mathcal{C}_3 : [a, b] \to \mathbb{R}^n$ looks forbidding. In the end, everything is just a matter of definition or what you want to calculate. If you look back on your courses in mathematics, the one-dimensional integration I_1 yields the area a function $f_1(x)$ forms with the x-axis and is defined as the limiting process of putting rectangles with width (x-direction) $\epsilon \to 0$ under the function to measure the area. See Fig. 2.1 for a graphical visualization of this. We will come to the other cases, like e.g. I_3, later.

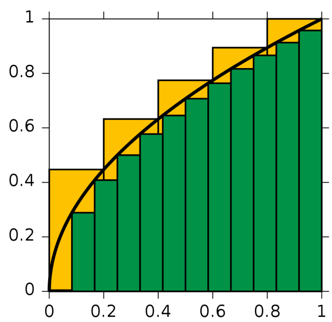

Figure 2.1: The limiting process of calculating the area under the function $f(x) = \sqrt{x}$ from 0 to 1 with rectangles. Image taken from [7].

2.2 Multi-Dimensional Integration

The next step would be to extend the domain of the function to a **higher-dimensional** vector space and keep the codomain's dimension constant. We get

$$\begin{aligned} f_2 : \quad & \mathbb{R}^2 \to \mathbb{R} \\ & (x, y) \mapsto f_2(x, y), \end{aligned}$$

which can be integrated over a domain $[a, b] \times [c, d] \subseteq \mathbb{R} \times \mathbb{R}$. The integral can then be referred to as a **double integral** and be written in the forms

$$I_2 = \int_a^b \mathrm{d}x \int_c^d \mathrm{d}y\, f_2(x, y) = \int_a^b \int_c^d \mathrm{d}x\, \mathrm{d}y\, f_2(x, y) = \int_{[a,b]\times[c,d]} \mathrm{d}x\, \mathrm{d}y\, f_2(x, y).$$

From now on, we usually write a vector for the function's values $\vec{x} = (x, y)$, which can then easily be extended to higher dimensions $\vec{x} = (x_1, x_2, \ldots)$ and the dimension of $\vec{x}$ will be clear from context. We will not make a difference between $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$, while strictly speaking, they are not the same. A Physicist would refer to the former one as a covariant and the latter one as a contravariant vector, sticking to tensor terminology. Mathematicians would refer to the latter one as a vector from some vector space $\mathbb{V}$, while the former one then is a vector from the dual space $\mathbb{V}^*$ of said vector space $\mathbb{V}$. The elements from a dual space may also be referred to as **one-forms**. Another notation that arises for I_2 then reads

$$I_2 = \int_{D_2} \mathrm{d}A\, f_2(\vec{x}) = \int_{D_2} \mathrm{d}^2x\, f_2(\vec{x}) = \int_{D_2} \mathrm{d}\vec{x}\, f_2(\vec{x})$$

with $D_2 = [a, b] \times [c, d]$, and $\mathrm{d}A = \mathrm{d}x\, \mathrm{d}y$ being an elementary infinitesimal piece of area. Again, mixtures of these notations are possible. A simple example is the integration of the constant function $f_2(\vec{x}) = 1$, yielding $I_2 = (b - a)(d - c)$, the area of the integration domain. A visualization of a more complicated function $f(x, y)$ is given in Fig. 2.2.

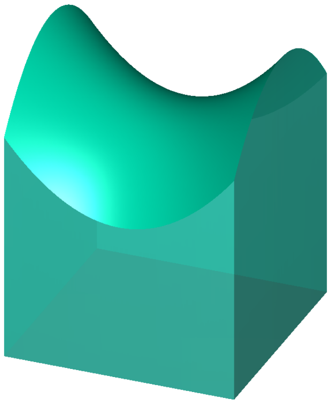

Figure 2.2: Some function $z = f(x, y)$ in a three-dimensional plot. The integral I_2 then yields the volume under the function $f(x, y)$. For $f(x, y) = 1$, one dimension is just the constant function, thus turning the volume into an area in some sense, because $V = A \times 1$ (1 in the z-direction). Image taken from [7].

2.3 Line Integrals

The next interesting thing to look at is the integration of a function over some line, resulting in a **line integral**. Going back to the one-dimensional case I_1, we integrated the function $y = f_1(x)$ over the x-axis. This is the easiest case: the domain of integration just lies along the x-axis and the function itself only takes scalar values. But it is also possible, that the domain of integration is some line in $\mathbb{R}^n$ and we want to integrate a function that takes on values in $\mathbb{R}^n$.

2.3.1 Line Integral of a Scalar Field

Let us first consider the case where we have a scalar field

$$f_3 : \mathbb{R}^n \to \mathbb{R}$$

and a curve $\mathcal{C}_3$, parametrized by

$$\mathcal{C}_3 : [a, b] \to \mathbb{R}^n.$$

The integration process of a function f along some curve $\mathcal{C}$ is shown and explained in Fig. 2.3 for the special case of $n = 2$. For the integral of the function f_3 along the curve $\mathcal{C}_3$, we generally define

$$I_3 = \int_{\mathcal{C}_3} \mathrm{d}s\, f_3(\vec{x}) = \int_a^b \mathrm{d}t\, f_3(\mathcal{C}_3(t)) \|\dot{\mathcal{C}}_3(t)\|,$$

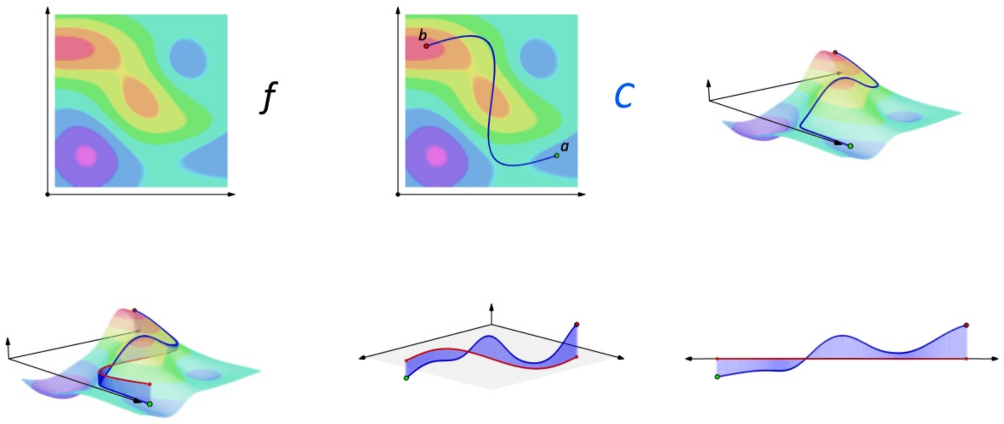

Figure 2.3: The integration of a scalar field f along some curve $\mathcal{C}$. The first image shows the contour plot of a function $f(x, y)$, the second image displays the integration curve $\mathcal{C}$ in blue color. On the third image, the contour plot turns into a three-dimensional plot, showing $f(x, y)$ on the z-axis. On the fourth image, the integration curve is projected onto the x-y-plane in red color. The fifth and sixth images show how the curve and its projection can be stretched to turn it into a one-dimensional integration. Images taken from the animation in [8].

with $\mathrm{d}s = \|\dot{\mathcal{C}}_3(t)\|\,\mathrm{d}t$ being an elementary infinitesimal piece of the curve. Furthermore, $C_3(t)$ is the parametrization of the curve C_3 with curve parameter $t \in [a, b]$, and

$$\|\dot{\mathcal{C}}_3(t)\| = \sqrt{(\dot{\mathcal{C}}_3(t)^1)^2 + (\dot{\mathcal{C}}_3(t)^2)^2 + \dots}$$

the euclidean norm. What this means is, that we integrate the function's values along the curve and weight these with the **arc length** $\|\dot{\mathcal{C}}_3(t)\|\,\mathrm{d}t$ at every point. An application of this would for example be the calculation of some quantity for a given density function of that quantity on the curve. To fully understand the appearance of the arc length, it might be useful to look into your favorite Analysis II literature for a motivation of this definition. Yet, it again turns out to be insightful to consider the case $f_3(\vec{x}) = 1$. In two dimensions, we then find

$$I_3 = \int_{\mathcal{C}_3} \mathrm{d}s = \int_a^b \mathrm{d}t\,\|\dot{\mathcal{C}}_3(t)\|,$$

which turns the integral into the length of the curve. This can easily be checked and for the special case of a unit circle centered at (x_0, y_0) with the parametrization

$$\begin{aligned}\mathcal{C}_3 :\ [0, 2\pi] &\to \mathcal{S}^1 \\ t &\mapsto \mathcal{C}_3(t) = (x_0 + \cos t, y_0 + \sin t)\end{aligned}$$

yields exactly what we expect, namely

$$I_3 = \int_0^{2\pi} \mathrm{d}t \, \|(-\sin t)^2 + (\cos t)^2\| = \int_0^{2\pi} \mathrm{d}t = 2\pi.$$

2.3.2 Line Integral of a Vector Field

In a next step, we want to extend the concept of line integrals to vector fields of the form

$$f_4 :\ \mathbb{R}^n \to \mathbb{R}^n$$

and curves $\mathcal{C}_4$, parametrized by

$$\mathcal{C}_4 :\ [a, b] \to \mathbb{R}^n.$$

It is important here, that the dimensions of the two codomains coincide. This becomes clear when looking at the definition of the integral of said vector field f_4 along the curve $\mathcal{C}_4$, given by

$$I_4 = \int_{\mathcal{C}_4} \mathrm{d}\vec{x} \cdot f_4(\vec{x}) = \int_a^b \mathrm{d}t \, f_4(\mathcal{C}_4(t)) \cdot \dot{\mathcal{C}}_4(t).$$

Here, $\mathrm{d}\vec{x}$ is a vector tangential to the curve with magnitude of the infinitesimal piece $\|\mathcal{C}_4(t)\| \, \mathrm{d}t$ as before and '$\cdot$' denotes the canonical scalar product on $\mathbb{R}^n$. The scalar product of course requires two vectors from the same vector space. But it is also pretty descriptive if one for example regards f_4 as a force and $\mathcal{C}_4$ as the curve along which the force acts. Because of the formula for work $\Delta W = \Delta F \, \Delta l$ with ΔF being the force and Δl the length along which it acts, I_4 represents the integration of the infinitesimal version of this formula. Of course, this makes only sense if the force acts in the same vector space as the way along which it acts. The integration procedure is depicted in Fig. 2.4.

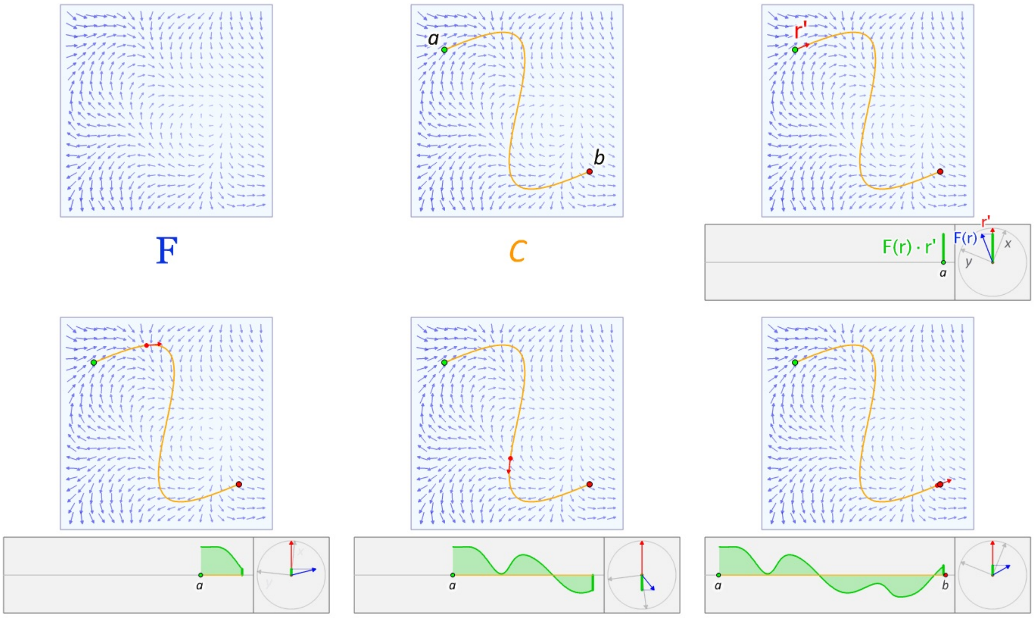

Figure 2.4: The integration of a vector field F along some curve $\mathcal{C}$. The images are self-explanatory. Images taken from the animation in [8].

2.4 Surface Integrals

Similarly to the motivation we gave for the line integrals, one can motivate **surface integrals**. The multi-dimensional integration I_2 can be extended to further cases. One is, where the domain of integration is not simply given by the cartesian product $D_2 = [a, b] \times [c, d]$, but rather a more complex 2-dimensional surface embedded in n-dimensional space (a **manifold**). The other is that additionally, the function itself might take values in $\mathbb{R}^n$. Even though we first define (two-dimensional) surface integrals for general n-dimensional spaces, we then turn to the case of $n = 3$ to find a motivation for the formula. Note further, that a **hypersurface** is not the same as a surface here. A hypersurface generalizes the concept of a surface to a $(n-1)$-dimensional manifold embedded in n-dimensional space. For $n = 3$, the concepts of surface and hypersurface coincide. In principle, it is more natural to talk about hypersurfaces than two-dimensional manifolds embedded in n-dimensional space. Just as it makes more sense to talk about a volume in n-dimensional space being an n-dimensional object itself, rather than a three-dimensional manifold embedded in n-dimensional space. If you want to learn more on the integration over hypersurfaces, you can for example take a look at [11].

2.4.1 Surface Integral of a Scalar Field

As before, we first consider the case of a scalar field

$$f_5 : \mathbb{R}^n \to \mathbb{R}$$

and a surface $\mathcal{A}_5$, parametrized by

$$\mathcal{A}_5 : [a,b] \times [c,d] \to \mathbb{R}^n.$$

What we want to calculate is the integral of the function f_5 over the surface $\mathcal{A}_5$, weighted with the infinitesimal surface element $\mathrm{d}S$ at every point. In Fig. 2.5, a depiction of this surface element $\mathrm{d}S$ can be found. Applications of this formula again include the calculation of some quantity with a given density function of that quantity on the surface. We define the integral of f_5 over the surface $\mathcal{A}_5$ to be

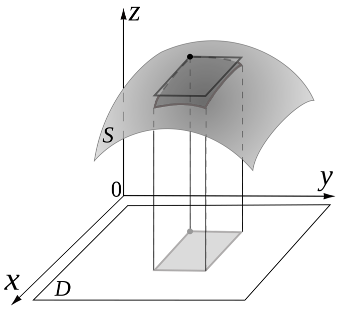

Figure 2.5: An infinitesimal small surface element (scaled up to a larger size) of a surface S for the case $n = 3$. Image taken from [9].

$$I_5 = \int_{\mathcal{A}_5} \mathrm{d}S\, f_5(\vec{x}) = \int_a^b \mathrm{d}t \int_c^d \mathrm{d}s\, f_5(\mathcal{A}_5(t,s))\sqrt{|\det(\sigma(t,s))|},$$

where

$$\sigma(t,s) = \begin{pmatrix} \frac{\partial \mathcal{A}_5(t,s)}{\partial t} \cdot \frac{\partial \mathcal{A}_5(t,s)}{\partial t} & \frac{\partial \mathcal{A}_5(t,s)}{\partial t} \cdot \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \\ \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \cdot \frac{\partial \mathcal{A}_5(t,s)}{\partial t} & \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \cdot \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \end{pmatrix}$$

and $\sqrt{\det(\sigma(t,s))}\,\mathrm{d}t\,\mathrm{d}s$ is the area of the surface element $\mathrm{d}S$ mentioned above. To understand this is beyond the scope of this recap and requires some knowledge on **differential geometry**. Working with the terminology of differential geometry, $\sigma(t,s)$ can be referred to as the **first fundamental form** of the parametrization for the surface in $n = 3$. What we will do, is look at this special case $n = 3$. Here, one easily calculates

$$\det(\sigma(t,s)) = \left\| \frac{\partial \mathcal{A}_5(t,s)}{\partial t} \right\|^2 \left\| \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \right\|^2 - \left(\frac{\partial \mathcal{A}_5(t,s)}{\partial t} \cdot \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \right)^2,$$

which, for the trained eye, obviously coincides with the square of the cross product's absolute value in three dimensions. To see this, we use

$$\epsilon_{kij}\epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

(sum convention), and find

$$\begin{aligned} \|a \times b\|^2 &= \epsilon_{kij} a_i b_j \epsilon_{kmn} a_m b_n = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) a_i b_j a_m b_n \\ &= \|a\|^2 \|b\|^2 - (a \cdot b)^2. \end{aligned}$$

Our integral can then be written as

$$I_5 = \int_{\mathcal{A}_5} \mathrm{d}S f_5(\vec{x}) = \int_a^b \mathrm{d}t \int_c^d \mathrm{d}s\, f_5(\mathcal{A}_5(t,s)) \left\| \frac{\partial \mathcal{A}_5(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \right\|,$$

where the area of the surface element $\mathrm{d}S$ is now given by $\left\| \frac{\partial \mathcal{A}_5(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_5(t,s)}{\partial s} \right\| \mathrm{d}t\, \mathrm{d}s$. This is exactly the infinitesimal form of the well-known formula to calculate the area spanned by two vectors a and b in three dimensions:

$$\|a \times b\| = \|a\| \|b\| \sin(\sphericalangle(a,b)).$$

As an example, we want to consider the integration of $f_5(\vec{x}) = 1$ over the unit sphere $\mathcal{S}^2$, centered at $(0,0,0)$. In spherical coordinates, this is given by the parametrization

$$\begin{aligned} \mathcal{A}_5 :\ & [0,\pi] \times [0,2\pi] \to \mathcal{S}^2 \\ & (\theta,\phi) \mapsto \mathcal{A}_5(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \end{aligned}$$

For the integral, we then find

$$\begin{aligned} I_5 &= \int_{\mathcal{A}_5} \mathrm{d}S = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \left\| \sin\theta (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \right\| \\ &= \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\, \sin\theta = 4\pi, \end{aligned}$$

as expected.

2.4.2 Surface Integral of a Vector Field

Having motivated the formula for the integration of a scalar field over a surface, we now come to the integration of a vector field

$$f_6 : \mathbb{R}^n \to \mathbb{R}^3$$

over a surface $\mathcal{A}_6$, parametrized by

$$\mathcal{A}_6 : [a, b] \times [c, d] \to \mathbb{R}^n.$$

Note how we restricted the codomain to a three-dimensional space. It is otherwise far from clear what is meant by this type of integration, because we would have to find an n-dimensional unit vector, normal to a two-dimensional surface. We usually have $n = 3$, and from a physical point of view, this integration represents the flux of the vector field through the area of the surface, as shown in Fig. 2.6. We define

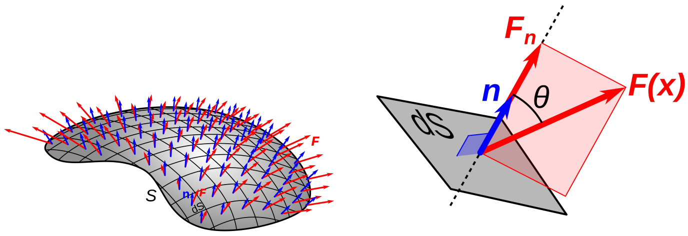

Figure 2.6: The vector field F, depicted in red color, flowing through the surface area. The blue vector field n represents the unit normal vectors on the surface. In the right image, it can be seen, how the actual flux through the surface is calculated by projecting the vector field at each point on the surface onto the unit vector thats normal to the surface at each point. Images taken from [9].

$$I_6 = \int_{\mathcal{A}_6} \mathrm{d}S \cdot f_6(\vec{x}),$$

with $\mathrm{d}S$ being a three-dimensional vector, normal to the surface at every point with the surface element's magnitude. This can then be rewritten as

$$I_6 = \int_{\mathcal{A}_6} \mathrm{d}S\, n(\vec{x}) \cdot f_6(\vec{x}) = \int_a^b \mathrm{d}t \int_c^d \mathrm{d}s\, n(\mathcal{A}_6(t,s)) \cdot f_6(\mathcal{A}_6(t,s)) \sqrt{\det(\sigma(t,s))},$$

where now, the scalar quantity $\mathrm{d}S = \sqrt{\det(\sigma(t,s))}\, \mathrm{d}t\, \mathrm{d}s$ is the area of the surface element like in the scalar case and $n(\mathcal{A}_6(t,s))$ represents a unit vector normal to the surface at every point. This is also where it gets problematic for the n-dimensional case. Yet, for the three-dimensional case $n = 3$, this simplifies to

$$\begin{aligned}
I_6 &= \int_{\mathcal{A}_6} \mathrm{d}S\, n(\mathcal{A}_6(t,s)) \cdot f_6(\vec{x}) \\
&= \int_a^b \mathrm{d}t \int_c^d \mathrm{d}s\, \frac{\left(\frac{\partial \mathcal{A}_6(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_6(t,s)}{\partial s}\right)}{\left\| \frac{\partial \mathcal{A}_6(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_6(t,s)}{\partial s} \right\|} \cdot f_6(\mathcal{A}_6(t,s)) \left\| \frac{\partial \mathcal{A}_6(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_6(t,s)}{\partial s} \right\| \\
&= \int_a^b \mathrm{d}t \int_c^d \mathrm{d}s \left(\frac{\partial \mathcal{A}_6(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_6(t,s)}{\partial s} \right) \cdot f_6(\mathcal{A}_6(t,s)),
\end{aligned}$$

because

$$n(\mathcal{A}_6(t,s)) = \frac{\left(\frac{\partial \mathcal{A}_6(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_6(t,s)}{\partial s}\right)}{\left\| \frac{\partial \mathcal{A}_6(t,s)}{\partial t} \times \frac{\partial \mathcal{A}_6(t,s)}{\partial s} \right\|}$$

is an appropriate unit vector, normal to the surface.

For the concept of a hypersurface (*reminder: $(n-1)$-dimensional manifold embedded in n-dimensional space*) on the other hand, the unit normal vector at every point is uniquely defined. The domain and codomain of the vector field that is to be integrated need to have the same dimension then, with the manifold being embedded in these and having one dimension less. We will not further talk about this here, but we want to mention, that the first fundamental form $\sigma(t,s)$ from above can easily be extended to higher-dimensional parametrizations $\mathcal{H}(t_1, t_2, \ldots)$ of a hypersurface $\mathcal{H}$, following

$$\Sigma(t_1, t_2, \ldots) = \begin{pmatrix} \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_1} \cdot \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_1} & \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_1} \cdot \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_2} & \cdots \\ \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_2} \cdot \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_1} & \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_2} \cdot \frac{\partial \mathcal{H}(t_1,t_2,\ldots)}{\partial t_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

This also explains, how one can integrate a scalar field over a hypersurface. Namely by modifying I_5 accordingly, i.e. adding the additional integrations and replacing $\mathcal{A}_5(t,s)$ by the parametrization of the hypersurface $\mathcal{H}(t_1, t_2, \ldots)$, as well as $\sigma(t,s)$ by $\Sigma(t_1, t_2, \ldots)$.

2.5 Volume Integrals

As a last type of integral, we consider **volume integrals**. We noticed earlier, that an integration over a hypersurface is more natural than an integration over a two-dimensional surface embedded in n-dimensional space. That's why now, we restrict ourselves to the case, where the domain of our function has the same dimension as the volume we integrate over. Similar to line integrals and surface integrals, the volume integrals represent a generalization of multi-dimensional integrals. Considering I_2, we could add a third integration variable, turning it into a **triple integral**. The domain of integration for the volume integrals can now be some volume in space, instead of segments along the different axes.

It then only makes sense, to consider the integration of a scalar field, because we can't find a unit vector thats normal to this volume in the considered dimension. We consider the function

$$f_7 : \mathbb{R}^3 \to \mathbb{R}$$

and a volume $\mathcal{V}_7$, parametrized by

$$\mathcal{V}_7 : [a, b] \times [c, d] \times [e, f] \to \mathbb{R}^3.$$

As mentioned earlier, the three-dimensional volume we integrate over is embedded in three-dimensional space now. We want to calculate the integral of the function f_7 over the volume $\mathcal{V}_7$, weighted with an infinitesimal volume element $\mathrm{d}V$ at every point. This can for example be applied to find some quantity (like the mass) for a given density function in this volume. We define the integral of f_7 over the volume $\mathcal{V}_7$ to be given by

$$I_7 = \int_{\mathcal{V}_7} \mathrm{d}V \, f_7(\vec{x}) = \int_a^b \mathrm{d}t \int_c^d \mathrm{d}s \int_e^f \mathrm{d}r \, f_7(\mathcal{V}_7(t, s, r)) |\det(D\mathcal{V}_7(t, s, r))|,$$

where $\det(D\mathcal{V}_7(t, s, r))$ is the determinant of the Jacobian matrix

$$(D\mathcal{V}_7(t, s, r))_{i\{t,s,r\}} = \frac{\partial \mathcal{V}_{7\,i}}{\partial \{t, s, r\}}.$$

You might be familiar with this under the name **transformation theorem** from your courses in Analysis. Note that from the determinant's definition, we find

$$(a \times b) \cdot c = \epsilon_{ijk} a_i b_j c_k = \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix},$$

whereby the absolute value of this triple product $|(a \times b) \cdot c|$ is known to give the size of the volume spanned by a, b, c. One then finds

$$|\det(D\mathcal{V}_7(t,s,r))| = \left| \left(\frac{\partial \mathcal{V}_7}{\partial t} \times \frac{\partial \mathcal{V}_7}{\partial s} \right) \cdot \frac{\partial \mathcal{V}_7}{\partial r} \right|,$$

explaining why $|\det(D\mathcal{V}_7(t,s,r))|$ can be thought of as the size of the infinitesimal volume element $\mathrm{d}V$.

This can be extended to volumes of higher-dimensional space by considering the parametrization of the volume $\mathcal{G}$, given by $\mathcal{G}(t_1, t_2, \ldots)$. The Jacobian matrix then reads

$$(D\mathcal{G}(t_1, t_2, \ldots))_{ij} = \frac{\partial \mathcal{G}_i}{\partial t_j}.$$

The integration over a three-dimensional 'volume' in a four-dimensional space on the other hand, can be done using the integration over a hypersurface. This is, because the 'volume' has one dimension less than the space it is embedded in.

3 Gauss' Theorem and Stokes' Theorem

The famous and very useful theorems by Gauss and Stokes connect volume integrals over a scalar field with surface integrals over a vector field, and surface integrals over a vector field with line integrals over a vector field respectively.

3.1 Gauss' Theorem

Gauss' **theorem** states, that for a vector field

$$f_8 : \mathbb{R}^n \to \mathbb{R}^n$$

and a volume $\mathcal{V}_8$ in the same space, we have

$$I_8 = \int_{\mathcal{V}_8} \mathrm{d}V \, \nabla \cdot f_8(\vec{x}) = \int_{\partial\mathcal{V}_8} \mathrm{d}S \, n(\vec{x}) \cdot f_8(\vec{x}),$$

where $(\nabla \cdot \quad)$ is the **divergence** operator and $\partial\mathcal{V}_8$ the boundary of the volume, a $(n-1)$-dimensional manifold. Here, the volume $\mathcal{V}_8$ needs to be compact, which according to Heine and Borel is equivalent to being closed and bounded if we consider $\mathbb{R}^n$. Furthermore, the function $f_8(\vec{x})$ has to be continuously differentiable. This becomes really important if one works with a function like $f_8(\vec{x}) = \frac{1}{\sqrt{x_1^2+\ldots+x_n^2}}$, exhibiting a singularity at the origin. It is not continuously differentiable in a neighborhood of the origin and choosing a compact volume around the origin, Gauss' theorem can thus not be applied. If you exclude the origin as a single point on the other hand, the function will be continuously differentiable but the volume is not compact anymore. We will usually consider the case of a three-dimensional vector field in three-dimensional space, i.e. $n = 3$. Then, in cartesian coordinates, the operator $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots)$ is given by $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$. Thinking of a vector field's divergence as a measure for the sources of field lines minus the sinks, you see a graphical representation of Gauss' theorem in Fig. 3.1.

3.2 Stokes' Theorem

As the general form of STOKES' **theorem** in n dimensions dives too much into differential geometry and differential forms, we only look at the case of a vector field

$$f_9 : \mathbb{R}^3 \to \mathbb{R}^3$$

and a two-dimensional area $\mathcal{A}_9$ that can be parametrized by two parameters. Here, $\mathcal{A}_9$ lies in an open subset of $\mathbb{R}^3$ and has to be bounded by a curve $\partial\mathcal{A}_9$. Furthermore, the parametrization has to be a diffeomorphism with a tangent space at each point and the function $f_9(\vec{x})$ has to be continuously differentiable. Descriptively speaking, the area can not be closed (like a sphere would for example be), because one has to integrate over the boundary of this area. In some sense it means that the area can not be a surface of some volume. STOKES' theorem then states that

$$I_9 = \int_{\mathcal{A}_9} \mathrm{d}S \cdot (\nabla \times f_9(\vec{x})) = \int_{\partial\mathcal{A}_9} \mathrm{d}\vec{x} \cdot f_9(\vec{x}),$$

where $(\nabla \times \quad)$ is also referred to as the **curl** of a vector field. An illustration is given in Fig. 3.2.

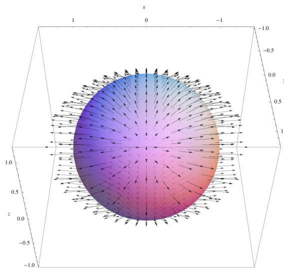

Figure 3.1: A graphical representation of a vector field flowing through the surface of a sphere. GAUSS' theorem relates this flow with the field lines that emerge and dissolve in the inner of the sphere, i.e. the volume. Image taken from [12].

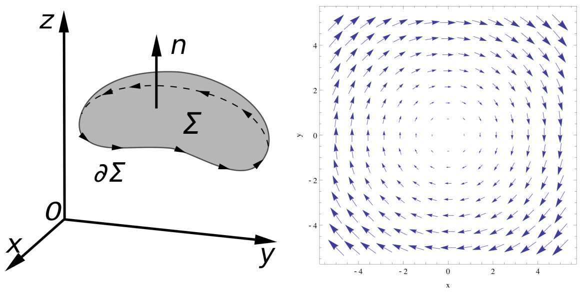

Figure 3.2: In the left image, you see the boundary of a surface area Σ, denoted as $\partial\Sigma$, and in the right image a rotating vector field. STOKES' theorem relates the integration of the curl of a vector field along an area with the line integral of the vector field along the boundary of that area. Images taken from [13] and [14].

4 Summary

We tried to motivate the existing formulae to calculate the integrals of certain types of function over different domains. If a motivation would have gone beyond the scope of this summary, we at least gave the formula to calculate the integrals. We covered line integrals, surface integrals and volume integral, as well as Gauss' theorem and Stokes' theorem. If you want to learn more on (**curved**) surfaces and some concepts of differential geometry, you can take courses on General Relativity and String Theory.

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