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Three-Body Decays 1

We want to calculate some additional steps with respect to the formulae for three-body decays in the PDG.

The decay width is given by

$$d\Gamma = \frac{(2\pi)^4}{2M} |M|^2 \, d\Phi_n(P; p_1, \dots, p_n) \,,$$

where M is the mass of the particle decaying into n particles and the phase space reads

$$d\Phi_n(P; p_1, \dots, p_n) = \delta^{(4)}\Big(P - \sum_{i=1}^n p_i\Big) \prod_{i=1}^n \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

The PDG then quotes that for a three-body decay, the decay width can be written as

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M} |M|^2 \, dE_1 \, dE_3 \, d\alpha \, d(\cos\beta) \, d\gamma \,, \qquad (*)$$

where α, β, γ are Euler angles, or

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M^2} |M|^2 \, |\vec{p}_1^{\,*}| \, |\vec{p}_3| \, dm_{12} \, d\Omega_1^* \, d\Omega_3 \,, \qquad (**)$$

where $(|\vec{p}_1^{\,*}|, \Omega_1^*)$ are the momentum and solid angle of particle 1 in the rest frame (CMS) of $(1,2)$ and $(|\vec{p}_3|, \Omega_3)$ the ones of particle 3 in the rest frame of the decaying particle. The latter formula can actually be derived from the recursion relation for the phase space; more specifically (in the PDG convention with factors of $(2\pi)^4$ etc.)

$$\begin{aligned}
d\Gamma &= \frac{(2\pi)^4}{2M} |M|^2 \, d\Phi_3(P; p_1, p_2, p_3) \\
&= \frac{(2\pi)^4}{2M} |M|^2 \, d\Phi_2(p_1+p_2; p_1, p_2) \, d\Phi_2(P; p_1+p_2, p_3) \, (2\pi)^3 \, dm_{12}^2 \\
&= \frac{1}{2M} |M|^2 \, \frac{1}{16\pi^2} \frac{|\vec{p}_1^{\,*}|}{E_{1,cm}^*} \, d\Omega_1^* \, \frac{1}{(2\pi)^4} \frac{1}{16\pi^2} \frac{|\vec{p}_3|}{E_{3,cm}} \, d\Omega_3 \, (2\pi)^3 \, dm_{12}^2
\end{aligned}$$

$$= \frac{1}{(2\pi)^5} \frac{1}{32M} \frac{|\vec{p}_1^{\,*}|}{E^*_{1,cm}} \frac{|\vec{p}_3|}{E_{3,cm}} |M|^2 \, dm_{12}^2 \, d\Omega_1^* \, d\Omega_3$$

$$\Big| \; \frac{dm_{12}^2}{dm_{12}} = 2m_{12}$$

$$= \frac{1}{(2\pi)^5} \frac{m_{12}}{16M} \frac{|\vec{p}_1^{\,*}|}{E^*_{1,cm}} \frac{|\vec{p}_3|}{E_{3,cm}} |M|^2 \, dm_{12} \, d\Omega_1^* \, d\Omega_3$$

| Note that $E_{3,cm} = M$, $E^*_{1,cm} = m_{12}$ (see also below)

$$= \frac{1}{(2\pi)^5} \frac{1}{16M^2} |\vec{p}_1^{\,*}| \, |\vec{p}_3| \, |M|^2 \, dm_{12} \, d\Omega_1^* \, d\Omega_3$$

For the momenta, we find (see mathematica)

$$|\vec{p}_1^{\,*}| = \frac{\sqrt{\lambda(m_{12}^2, m_1^2, m_2^2)}}{2m_{12}}, \quad E_1^* = \frac{m_{12}^2 + m_1^2 - m_2^2}{2m_{12}},$$

$$E_2^* = \frac{m_{12}^2 - m_1^2 + m_2^2}{2 \cdot m_{12}},$$

$$|\vec{p}_3| = \frac{\sqrt{\lambda(m_{12}^2, M^2, m_3^2)}}{2M}, \quad E_3 = \frac{M^2 - m_{12}^2 + m_3^2}{2M},$$

$$E_{12} = \frac{M^2 + m_{12}^2 - m_3^2}{2M},$$

$$\left(E_1^* + E_2^* = E^*_{1,cm} = m_{12} \; ; \; E_3 + E_{12} = E_{3,cm} = M \right).$$

Furthermore, we will need the energy of particle 3 in the rest frame (cms) of (1,2),

$$|\vec{p}_3^{\,*}| = \frac{\sqrt{\lambda(m_{12}^2, M^2, m_3^2)}}{2m_{12}}, \quad E_3^* = \frac{M^2 - m_{12}^2 - m_3^2}{2m_{12}}.$$

When only interested in spin- and polarization-averaged quantities, we can perform the angular integrations in (*), resulting in

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{8M} \overline{|M|^2} \, dE_1 \, dE_3$$

| $m_{12}^2 = (p_1+p_2)^2 = (P-p_3)^2 = M^2 + m_3^2 - 2ME_3$, $m_{23}^2 = (p_2+p_3)^2 = (P-p_1)^2 = M^2 + m_1^2 - 2ME_1$

| $m_{13}^2 = (p_1+p_3)^2 = (P-p_2)^2 = M^2 + m_2^2 - 2ME_2$

$$= \frac{1}{(2\pi)^3} \frac{1}{32M^3} \overline{|M|^2} \, dm_{12}^2 \, dm_{23}^2$$

Three-Body Decays 2

Note that the same formula can be obtained by using (**) instead: integrating over all angles except for one polar angle (which we will choose as the angle between $\vec{p}_1^{\,*}$ and $\vec{p}_3$), we find that

$$(**) \mathrel{\hat{=}} d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M^2} |\mathcal{M}|^2 |\vec{p}_1^{\,*}| |\vec{p}_3| \, dm_{12} \, d\Omega_1^* \, d\Omega_3$$

$$= \frac{1}{(2\pi)^3} \frac{1}{8M^2} \overline{|\mathcal{M}|^2} |\vec{p}_1^{\,*}| |\vec{p}_3| \frac{dm_{12}^2}{2m_{12}} d\cos\theta$$

$$m_{23}^2 = (E_2^* + E_3^*)^2 - (\vec{p}_2^{\,*} + \vec{p}_3^{\,*})^2$$

$$= (E_2^* + E_3^*)^2 - \left(|\vec{p}_2^{\,*}|^2 + |\vec{p}_3^{\,*}|^2 + 2|\vec{p}_2^{\,*}||\vec{p}_3^{\,*}| \cos\theta\right)$$

$$\rightsquigarrow \frac{dm_{23}^2}{d\cos\theta} = -2|\vec{p}_2^{\,*}||\vec{p}_3^{\,*}| = -2|\vec{p}_1^{\,*}||\vec{p}_3^{\,*}|$$

$$= -\frac{\sqrt{\lambda(m_{12}^2, m_1^2, m_2^2)} \sqrt{\lambda(m_{12}^2, M^2, m_3^2)}}{2m_{12}^2}$$

$$= -|\vec{p}_1^{\,*}||\vec{p}_3| \cdot \left(2\frac{M}{m_{12}}\right)$$

$$= \frac{1}{(2\pi)^3} \frac{\overline{|\mathcal{M}|^2}}{32M^3} dm_{12}^2 \, dm_{23}^2$$

For a given value of m_{12}^2, the domain of m_{23}^2 is then further determined by its values when $\vec{p}_2$ and $\vec{p}_3$ are parallel / anti-parallel, i.e. ($\cos\theta = \pm 1$)

$$(m_{23}^2)_{min} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2}\right)^2,$$

$$(m_{23}^2)_{max} = (E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2}\right)^2.$$