Disclaimer

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https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

26.04.2017 Quantum Field theory Exercise Sheet 1 — Marvin Zanke

H1)
$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) \quad \text{with} \quad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$
$$\overset{(*)}{=} -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x)$$

a) Euler-Lagrange equations, variation in $A_\mu(x)$

What is $\frac{\delta h}{\delta(\dots)}$? Not a regular derivative? generalisation → you can look into functional derivatives

works the same

$$0 = \partial_\lambda \frac{\delta \mathcal{L}}{\delta(\partial_\lambda A_k)} - \underbrace{\frac{\delta h}{\delta A_k}}$$

$= 0$, as $F_{\mu\nu}(x)$ doesn't depend on A_k explicitly!

Do we get $k=0,\dots,3$ equations here? ✓

using $(*)$
$$= \partial_\lambda \Big\{ -\frac{1}{4} \delta_{\mu\lambda}\delta_{\nu k} (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x))$$
$$+ \frac{1}{4} \delta_{\nu\lambda}\delta_{\mu k} (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x))$$
$$- \frac{1}{4} \delta_{\mu\lambda}\delta_{\nu k} (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x))$$
$$+ \frac{1}{4} \delta_{\nu\lambda}\delta_{\mu k} (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)) \Big\}$$

(alternatively using the metric tensor twice $\eta^{\nu\varepsilon}\eta^{\mu\delta}$)

$$= \frac{1}{4}\partial_\lambda \Big\{ -\partial^\lambda A^k(x) + \partial^k A^\lambda(x) + \partial^k A^\lambda(x) - \partial^\lambda A^k(x)$$
$$- \partial^\lambda A^k(x) + \partial^k A^\lambda(x) + \partial^k A^\lambda(x) - \partial^\lambda A^k(x) \Big\}$$

$$\overset{\substack{\partial_\mu\partial_\nu = \partial_\nu\partial_\mu \\ \text{Schwarz}}}{=} \partial_\lambda \left(\partial^k A^\lambda(x) - \partial^\lambda A^k(x) \right) = \partial^k(\partial A(x)) - \partial^2 (A^k(x))$$

where $\partial f := \partial_\mu f^\mu$ ✓

b)
$$\partial_\mu F^{\mu\nu}(x) \underset{\partial_\mu\partial_\nu = \partial_\nu\partial_\mu}{=} \partial_\mu \{ \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \} = \partial^2 A^\nu(x) - \partial^\nu(\partial A)(x) \overset{a)}{=} 0$$ ✓

With $\tilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\lambda\xi} F_{\lambda\xi}(x)$, we get

$$\partial_\mu \tilde{F}^{\mu\nu}(x) = \partial_\mu \left\{ \frac{1}{2} \epsilon^{\mu\nu\lambda\xi} F_{\lambda\xi}(x) \right\} = \frac{1}{2}\epsilon^{\mu\nu\lambda\xi} \partial_\mu \{ \partial_\lambda A_\xi(x) - \partial_\xi A_\lambda(x) \}$$

$$= \frac{1}{4}\epsilon^{\mu\nu\lambda\xi} \partial_\mu\partial_\lambda A_\xi(x) + \frac{1}{4} \underbrace{\epsilon^{\lambda\nu\mu\xi}}_{\text{antisym.}} \underbrace{\partial_\lambda\partial_\mu}_{\text{symmetric}} A_\xi(x)$$
$$- \frac{1}{4}\epsilon^{\mu\nu\lambda\xi} \partial_\mu\partial_\xi A_\lambda(x) - \frac{1}{4} \underbrace{\epsilon^{\xi\nu\lambda\mu}}_{\text{antisym.}} \underbrace{\partial_\xi\partial_\mu}_{\text{sym.}} A_\lambda(x)$$

Didn't use field equations here? true

$$= 0$$ ✓

c) $\mathcal{L}_{tot} = D_\mu \phi^*(x) D^\mu \phi(x) - m^2 \phi^*(x)\phi(x) - \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)$

where is this $\mathcal{L}_{tot}$ from?

with D_μ only acting on $\phi^*(x)$ etc.

What does it mean?

We already know this from a)

$= \{(\partial_\mu - ieA_\mu(x))\phi^*(x)\}\{(\partial^\mu + ieA^\mu(x))\phi(x)\} - m^2\phi^*(x)\phi(x) - \underbrace{\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)}$

ask me again in class. Answer is to long to write down.

Varying with respect to $A_\mu(x)$ once more yields:

$$0 = \partial_\lambda \frac{\delta \mathcal{L}}{\delta(\partial_\lambda A_k)} - \frac{\delta \mathcal{L}}{\delta A_k}$$

again using $D_\mu\phi^*(x)D^\mu\phi(x) = D^\mu\phi^*(x)D_\mu\phi(x)$

$$= \underbrace{\partial_\lambda\{-\partial^\lambda A^k(x) + \partial^k A^\lambda(x)\}}_{\text{from a), no new } (\partial_\lambda A_k)(x) \text{ dependence!}} - \{\delta_{\mu k}(-ie)\phi^*(x)D^\mu\phi(x) + \delta_{\mu k}(ie)D_\mu\phi^*(x)\phi(x)\}$$

$$= \underbrace{\partial^k(\partial A(x)) - \partial^2 A^k(x)}_{=\partial_\lambda F^{k\lambda}(x)} - ie\{(D_k\phi^*(x))\phi(x) - \phi^*(x)(D^k\phi(x))\}$$

Pull $\phi(x)$ to the front?

What is the question? Commute here!

$$\Longleftrightarrow \partial_\lambda F^{k\lambda}(x) = ie\{\phi(x) D_k\phi^*(x) - \phi^*(x)D^k\phi(x)\} =: -eJ^k(x)$$

using $J^k(x) := i(\phi^*(x)D^k\phi(x) - \phi(x)D^k\phi^*(x))$

d) $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + \frac{m^2}{2}B_\mu(x)B^\mu(x)$ with $F_{\mu\nu}(x) = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)$

What is this massive vector field?

$$0 = \partial_\lambda \frac{\delta \mathcal{L}}{\delta(\partial_\lambda B_k)} - \frac{\delta \mathcal{L}}{\delta B_k} = \underbrace{-\partial^2 B^k(x) + \partial^k(\partial B(x))}_{\text{from a), analogue}} - \frac{m^2}{2}(B^k \cdot 2)$$

$$\Longleftrightarrow \partial^2 B^k(x) - \partial^k(\partial B(x)) + m^2 B^k(x) = 0 \checkmark$$

so far it's just a toy-model an example for a massive ~~Massi~~ Vector-Field would be the Gauge Bosons of SU(2) $W^\pm_\mu$ Z_μ

e) Applying ∂_k to both sides of d) (which is a sum of derivatives of the same equation in the end) yields:

$$\overset{d)}{\Longleftrightarrow} \partial_k\{\partial^2 B^k(x) - \partial^k(\partial B(x)) + m^2 B^k(x)\} = 0$$

$$\Longleftrightarrow \partial^2(\partial B(x)) - \partial^2(\partial B(x)) + m^2\partial_k B^k(x) = 0 = \underline{\underline{m^2 \partial_k B^k(x)}}$$

$$\Longleftrightarrow \partial_k B^k(x) = 0 \checkmark$$

$\rightsquigarrow$ using d), we get $\partial^2 B^k(x) - \partial^k(\partial B(x)) + m^2 B^k(x) = 0$

$$= \partial^2 B^k(x) + m^2 B^k(x) = (\partial^2 + m^2)B^k(x) \checkmark$$