Disclaimer

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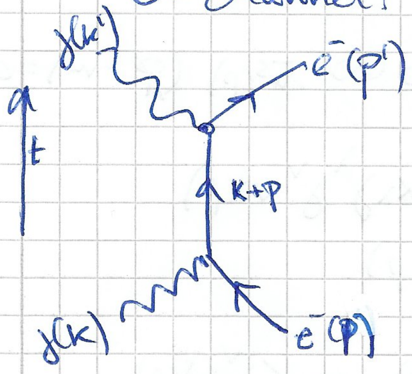
I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

12.07.2017 Quantum Field Theory Exercise sheet 10 Marvin Zanke

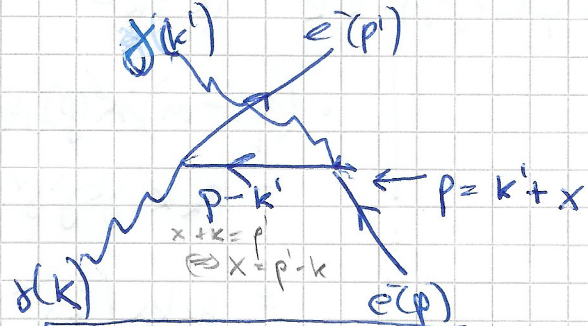
H1)
$$\gamma(k)\, e^-(p) \longrightarrow \gamma(k')\, e^-(p')$$
↑ mass m

a)
$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu$$

s-channel: 2nd order u-channel:



In general:
$k+p = k'+p'$



$$\propto \langle \gamma e^- | \bar{\psi}_x \gamma^\mu_x \psi_x A_{\mu,x}\, \bar{\psi}_y \gamma^\nu_y \psi_y A_{\nu,y} | \gamma e^- \rangle \qquad \langle \gamma e^- | \bar{\psi}_x \gamma^\mu \psi_x A_{\mu,x}\, \bar{\psi}_y \gamma^\nu_y \psi_y A_{\nu,y} | \gamma e^- \rangle$$

⇝ No add. sign ⇝ No add. sign

$$iM_1 = \bar{u}^{s'}(p')(-ie\gamma^\mu)\left(\epsilon_\mu^{\lambda'}(k')\right)^* \frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2} \epsilon_\nu^\lambda(k)(-ie\gamma^\nu)u^s(p)$$

$$iM_2 = \bar{u}^{s'}(p')(-ie\gamma^\mu)\epsilon_\mu^\lambda(k) \frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2} \left(\epsilon_\nu^{\lambda'}(k')\right)^*(-ie\gamma^\nu)u^s(p)$$

$$\rightsquigarrow iM = \bar{u}^{s'}(p')\left(\epsilon_\mu^{\lambda'}(k')\right)^*(-ie^2)\epsilon_\nu^\lambda(k)$$
$$\times\left\{\gamma^\mu \frac{(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\gamma^\nu + \gamma^\nu \frac{(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}\gamma^\mu\right\}u^s(p)$$

↑ $k^2+p^2+2kp = m^2+2kp$ ↑ $p^2+k'^2-2k'p = m^2-2k'p$

Now: Dirac eq. for spinors $(\not{p}-m)u^s(p)=0$; similarly want to calculate

$$(\not{p}+m)\gamma^\nu u^s(p) = (p_\kappa\gamma^\kappa+m)\gamma^\nu u^s(p) = p_\kappa(2g^{\kappa\nu}-\gamma^\nu\gamma^\kappa)u^s(p) + m\gamma^\nu u^s(p)$$
$$= 2p^\nu u^s(p) \underbrace{-\gamma^\nu \not{p}u^s(p) + m\gamma^\nu u^s(p)}_{-\gamma^\nu(\not{p}-m)u^s(p)=0} = 2p^\nu u^s(p)$$

$$\rightsquigarrow M = -e^2\left(\epsilon_\mu^{\lambda'}(k')\right)^*\epsilon_\nu^\lambda(k)\bar{u}^{s'}(p')\left\{\gamma^\mu\frac{\not{k}\gamma^\nu+2p^\nu}{2(k\cdot p)} + \gamma^\nu\frac{2p^\mu-\not{k}'\gamma^\mu}{-2k'\cdot p}\right\}u^s(p)$$ ✓

$$= -e^2\left(\epsilon_\mu^{\lambda'}(k')\right)^*\epsilon_\nu^\lambda(k)\bar{u}^{s'}(p')\left\{\frac{\gamma^\mu\not{k}\gamma^\nu+2\gamma^\mu p^\nu}{2(k\cdot p)} - \frac{-\gamma^\nu\not{k}'\gamma^\mu+2\gamma^\nu p^\mu}{2(k'\cdot p)}\right\}u^s(p)$$

b) $k'_\mu M^{\mu\nu} = \bar{u}^{s'}(p') \left\{ \frac{\not{k}'\not{k}\gamma^\nu + 2\not{k}'p^\nu}{2(k\cdot p)} - \frac{-\gamma^\nu \not{k}'\not{k}' + 2\gamma^\nu (k'\cdot p)}{2(k'\cdot p)} \right\} u^s(p)$

$k' = k + p - p'$

$= \bar{u}^{s'}(p') \left\{ \frac{(\not{k} + \not{p} - \not{p}')\not{k}\gamma^\nu + 2(\not{k} + \not{p} - \not{p}')p^\nu}{2(k\cdot p)} - \frac{\gamma^\nu (k'\cdot k') + 2\gamma^\nu (k'\cdot p)}{2(k'\cdot p)} \right\} u^s(p)$

$\not{k}\cdot\not{k} = \frac{1}{2}(k_\mu \gamma^\mu k_\nu \gamma^\nu + k_\nu \gamma^\nu k_\mu \gamma^\mu) = \frac{1}{2} k_\mu k_\nu \{\gamma^\mu, \gamma^\nu\} = k\cdot k$

$(k^{(\prime)})^2 = 0 \qquad (\not{p} - m) u^s(p) = 0 \Leftrightarrow \bar{u}^s(p)(\not{p} - m) = 0$

$\cdot\ (\not{p} - m)\not{k}\gamma^\nu u^s(p) = (p_\kappa \gamma^\kappa - m) k_\lambda \gamma^\lambda \gamma^\nu u^s(p)$

$= (p_\kappa k_\lambda \gamma^\kappa \gamma^\lambda \gamma^\nu - m k_\lambda \gamma^\lambda \gamma^\nu) u^s(p)$

$= p_\kappa k_\lambda (2g^{\lambda\kappa}\gamma^\nu - \gamma^\lambda \gamma^\kappa \gamma^\nu) u_s(p) - m\not{k}\gamma^\nu u^s(p)$

$= p_\kappa k_\lambda (2g^{\lambda\kappa}\gamma^\nu - 2g^{\kappa\nu}\gamma^\lambda + \gamma^\lambda \gamma^\nu \gamma^\kappa) u_s(p) - m\not{k}\gamma^\nu u^s(p)$

$= (2(p\cdot k)\gamma^\nu - 2\not{k}p^\nu + \not{k}\gamma^\nu \not{p}) u_s(p) - m\not{k}\gamma^\nu u^s(p)$ ($\not{p}\, u_s(p) = m u^s(p)$)

$= (2(p\cdot k)\gamma^\nu - 2\not{k}p^\nu) u^s(p)$

$= \bar{u}^{s'}(p') \left\{ \frac{2(p\cdot k)\gamma^\nu - 2\not{k}p^\nu + m\not{k}\gamma^\nu - m\not{k}\gamma^\nu + 2\not{k}p^\nu + 2(m-m)p^\nu}{2(k\cdot p)} - \frac{2\gamma^\nu (k'\cdot p)}{2(k'\cdot p)} \right\} u^s(p)$

$= \bar{u}^{s'}(p') \{ \gamma^\nu - \gamma^\nu \} u^s(p) = 0$ ✓

c) Lösung aus dem Tutorium !

$$\epsilon^{(0)}(k) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{|\vec{k}|}\begin{pmatrix}|\vec{k}|\\0\\0\\0\end{pmatrix} \text{ (scalar)}, \quad \epsilon^{(3)}(k) = \frac{1}{|\vec{k}|}\begin{pmatrix}0\\\vec{k}\end{pmatrix} \text{ (longitudinal)}, \quad \text{e.g. } \epsilon^{1,2} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix}$$

$$\epsilon^{(0)}_\mu + \epsilon^{(3)}_\mu = \frac{k_\mu}{|\vec{k}|}, \quad 0 = k_\mu' \epsilon_\nu^{(\lambda)} M^{\mu\nu} = \left(\epsilon_\mu^{(0)}(k') + \epsilon_\mu^{(3)}(k')\right)\epsilon_\nu^{(\lambda)} M^{\mu\nu} \cdot |\vec{k}'|$$

$$\rightsquigarrow \epsilon_\mu^0(k')\epsilon_\nu^{(\lambda)} M^{\mu\nu} = -\epsilon_\mu^3(k')\epsilon_\nu^{(\lambda)} M^{\mu\nu} \quad (*)$$

$$\left(\epsilon_\mu^{(\lambda')}(k')\right)^* \epsilon_\nu^0 M^{\mu\nu} = -\left(\epsilon_\mu^{\lambda'}(k')\right)^* \epsilon_\nu^3 M^{\mu\nu} \quad (**)$$

↑ from $k_\nu M^{\mu\nu} = 0$ analogue
$k_\nu \epsilon_\mu^{(\lambda')} M^{\mu\nu}$
$\left(\epsilon_\nu^0(k) + \epsilon_\nu^3(k)\right)|\vec{k}|\epsilon_\mu^{(\lambda')} M^{\mu\nu}$

$$\frac{1}{2}\sum_{\lambda,\lambda'=0}^{3}(-g_{\lambda\lambda})(-g_{\lambda'\lambda'})\left(\epsilon_{\mu'}^{\lambda' *}\epsilon_{\nu'}^{\lambda} M^{\mu'\nu'}\right)^* \left(\epsilon_\mu^{*\lambda'}\epsilon_\nu^{\lambda} M^{\mu\nu}\right)$$

$$= \frac{1}{2}\sum_{\lambda,\lambda'=0}^{3}(-g_{\lambda\lambda})(-g_{\lambda'\lambda'})\left|\epsilon_{\mu'}^{*\lambda'}\epsilon_{\nu'}^{\lambda} M^{\mu'\nu'}\right|^2$$

$$= \frac{1}{2}\sum_{\lambda,\lambda'=1}^{2}\left|\epsilon_{\mu'}^{*\lambda'}\epsilon_{\nu'}^{\lambda} M^{\mu'\nu'}\right|^2 + \frac{1}{2}\sum_{\lambda=0}^{3} g_{\lambda\lambda}\underbrace{\left|\epsilon_{\mu'}^{*0}\epsilon_{\nu'}^{\lambda} M^{\mu'\nu'}\right|^2}_{\text{due to } (*)}$$

$$- \frac{1}{2}\sum_{\lambda=0}^{3} g_{\lambda\lambda}\left|\epsilon_{\mu'}^{*3}\epsilon_{\nu'}^{\lambda} M^{\mu'\nu'}\right|^2$$

$$+ \frac{1}{2}\sum_{\lambda'=1}^{2} g_{\lambda'\lambda'}\underbrace{\left|\epsilon_{\mu'}^{*\lambda'}\epsilon_{\nu'}^{0} M^{\mu'\nu'}\right|^2}_{\text{due to } (**)}$$

$$- \frac{1}{2}\sum_{\lambda'=1}^{2} g_{\lambda'\lambda'}\left|\epsilon_{\mu'}^{*\lambda'}\epsilon_{\nu'}^{3} M^{\mu'\nu'}\right|^2$$

d)

$$\overline{|M|^2} = \frac{1}{2}\frac{1}{2}\sum_{s,s'}\sum_{\lambda,\lambda'=0}^{3} e^4 (-g_{\lambda\lambda})(-g_{\lambda'\lambda'})\, M^{\mu\nu}\left(M^{\mu'\nu'}\right)^* \left(\epsilon_\mu^{\lambda'}(k')\right)^* \epsilon_{\mu'}^{\lambda'}(k')\, \epsilon_\nu^{\lambda}(k)\left(\epsilon_{\nu'}^{\lambda}(k)\right)^*$$

$$= \frac{e^4}{4}\sum_{s,s'} g_{\mu\mu'} g_{\nu\nu'} M^{\mu\nu}\left(M^{\mu'\nu'}\right)^* \qquad \leftarrow \left(M^{\mu'\nu'}\right)^* = \left(M^{\mu'\nu'}\right)^\dagger \text{, as scalar}$$

$$\left(M^{\mu'\nu'}\right)^\dagger = u^{\dagger s}(p)\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2(k\cdot p)} - \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{2(k'\cdot p)}\right]^\dagger (\gamma^0)^\dagger u^{s'}(p')$$

$$\not{k}^\dagger = (k_\mu\gamma^\mu)^\dagger = \gamma^0\not{k}\gamma^0$$

$$= u^{\dagger s}(p)\left[\frac{\gamma^0\gamma^\nu\gamma^0\gamma^0\not{k}\gamma^0\gamma^0\gamma^\mu\gamma^0 + 2\gamma^0\gamma^\mu\gamma^0 p^\nu}{2(k\cdot p)} - \frac{\ldots + 2\gamma^0\gamma^\nu\gamma^0 p^\mu}{2(k'\cdot p)}\right]\gamma^0 u^{s'}(p')$$

$$= \bar{u}^s(p)\left[\frac{\gamma^\nu\not{k}\gamma^\mu + 2\gamma^\mu p^\nu}{2(k\cdot p)} - \frac{-\gamma^\mu\not{k}'\gamma^\nu + 2\gamma^\nu p^\mu}{2(k'\cdot p)}\right] u^{s'}(p')$$

$$= \frac{e^4}{4}\sum_{s,s'} \bar{u}^{s'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2(k\cdot p)} - \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{2(k'\cdot p)}\right] u^s(p)$$

$$\times\, \bar{u}^s(p)\left[\frac{\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu}{2(k\cdot p)} - \frac{-\gamma_\mu\not{k}'\gamma_\nu + 2\gamma_\nu p_\mu}{2(k'\cdot p)}\right] u^{s'}(p')$$

Tr(…) and shift $u^{s'}(p')$ to front

$$= \frac{e^4}{4}\sum_{s,s'} \mathrm{Tr}\left\{ u^{s'}(p')\bar{u}^{s'}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2(k\cdot p)} - \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{2(k'\cdot p)}\right]\right.$$

$$\left.\times\, u^s(p)\bar{u}^s(p)\left[\frac{\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu}{2(k\cdot p)} - \frac{-\gamma_\mu\not{k}'\gamma_\nu + 2\gamma_\nu p_\mu}{2(k'\cdot p)}\right]\right\}$$

$$= \frac{e^4}{4}\mathrm{Tr}\left\{(\not{p}'+m)\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2(k\cdot p)} - \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{2(k'\cdot p)}\right]\right.$$

$$\left.\times\,(\not{p}+m)\left[\frac{\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu}{2(k\cdot p)} - \frac{-\gamma_\mu\not{k}'\gamma_\nu + 2\gamma_\nu p_\mu}{2(k'\cdot p)}\right]\right\}$$

$$= \frac{e^4}{4}\left\{\mathrm{Tr}\left\{(\not{p}'+m)\frac{(\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p}+m)(\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu)}{(2(k\cdot p))^2}\right\}\right.$$

$$+\,\mathrm{Tr}\left\{(\not{p}'+m)\frac{(-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu)(\not{p}+m)(-\gamma_\mu\not{k}'\gamma_\nu + 2\gamma_\nu p_\mu)}{(2(k'\cdot p))^2}\right\}$$

$$+\,\mathrm{Tr}\left\{(\not{p}'+m)\frac{(\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p}+m)(\gamma_\mu\not{k}'\gamma_\nu - 2\gamma_\nu p_\mu)}{2(k\cdot p)\,2(k'\cdot p)}\right\}$$

$$\left.+\,\mathrm{Tr}\left\{(\not{p}'+m)\frac{(\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu)(\not{p}+m)(\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu)}{2(k\cdot p)\,2(k'\cdot p)}\right\}\right\}$$

$\nu \leftrightarrow \mu$ for 2nd

$$= \frac{e^4}{4}\left\{\frac{I_1}{(2(k\cdot p))^2} + \frac{I_2}{(2(k'\cdot p))^2} + \frac{I_3 + I_4}{2(k\cdot p)\,2(k'\cdot p)}\right\}$$

Where did λ,λ' go? ✓ are $= \bar{s}$

$s, s' = 1,2$ ✓ or $\pm 1/2$

Sum only over $\lambda,\lambda' = 1,2$? why?

Same λ', different p'?

$\epsilon_\nu(k\bar{s})$? ✓ $\bar{s} = \lambda,\lambda'$

No trace trick for $\Sigma_{\lambda\lambda'}$?

$I_3 \overset{?}{=} I_4$ need to show?

e) $I_1 = Tr\{(\not p' + m)[\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu](\not p + m)[\gamma_\nu \not k \gamma_\mu + 2\gamma_\mu p_\nu]\}$

Tr(odd) vanishes $\Rightarrow Tr\{\not p'[\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu]\not p[\gamma_\nu \not k \gamma_\mu + 2\gamma_\mu p_\nu]$
$+ m^2[\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu][\gamma_\nu \not k \gamma_\mu + 2\gamma_\mu p_\nu]\}$

Only w/ help of p.p?

$= Tr\{\not p'\gamma^\mu \not k \gamma^\nu \not p \gamma_\nu \not k \gamma_\mu + 2\not p'\gamma^\mu \not k \gamma^\nu \not p \gamma_\mu p_\nu + 2\not p'\gamma^\mu p^\nu \not p \gamma_\nu \not k \gamma_\mu + 4\not p'\gamma^\mu p^\nu \not p \gamma_\mu p_\nu$ (with $-2\not p$, $\not p$, $\not p$, p^2 above; $-2\not k\not p\not k$, $-2\not p\not p\not k$, $-2\not k\not p\not p$, $-2\not p$ below)

$+ m^2 Tr\{\gamma^\mu \not k \gamma^\nu \gamma_\nu \not k \gamma_\mu + 2\gamma^\mu \not k \gamma^\nu \gamma_\mu p_\nu + 2\gamma^\mu p^\nu \gamma_\nu \not k \gamma_\mu + 4\gamma^\mu p^\nu \gamma_\mu p_\nu\}$ (with 4, $4(k\cdot k)$; $\not p$, $4(k\cdot p)$; $\not p$, $4(p\cdot k)$; 4, p^2 below)

Where from $p\cdot p' = pk - pk' + m^2$

$\not p^2 = p_\mu \gamma^\mu p_\nu \gamma^\nu = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = p\cdot p$

$\Rightarrow Tr\{4\not p'\not k \not p \not k - 4\not p'\not p\not p\not k - 4\not p'\not k\not p\not p - 8\not p'\not p\, p^2\}$

$+ m^2 Tr\{16k^2 + 8(k\cdot p) + 8(p\cdot k) + 16p^2\}$

$= 16\{[(p'\cdot k)(p\cdot k) - (p'\cdot p)(k\cdot k) + (p'\cdot k)(k\cdot p)]$
$- [(p'\cdot p)(p\cdot k) - (p'\cdot p)(p\cdot k) + (p'\cdot k)(p\cdot p)]$
$- [(p'\cdot k)(p\cdot p) - (p'\cdot p)(k\cdot p) + (p'\cdot p)(k\cdot p)]\} - 8p^2\cdot(4p'\cdot p)$

$+ 4m^2\{16k^2 + 16(k\cdot p) + 16p^2\}$

$p'\cdot p = p\cdot k - p\cdot k' + m^2$
$= 16\{2(p'\cdot k)(p\cdot k)\} - 32m^2((p\cdot k) - (p\cdot k') + m^2) - 32m^2(p\cdot k')$
$+ 64m^2(k\cdot p) + 64m^4$

What if not possible to get totally rid of p'?

$p'\cdot k = p\cdot k'$
$= 16\{2(p\cdot k')(p\cdot k) - 2m^2(p\cdot k) + 2m^2(p\cdot k') - 2m^4$
$+ 4m^2(k\cdot p) + 4m^4 - 2m^2(p\cdot k')\}$

$= 16\{2m^4 + 2m^2(p\cdot k) + 2(p\cdot k')(p\cdot k)\}$

it is not necessary to get rid of p' it's just nice.

Tutorial: $Tr[\gamma^\mu \not p \gamma_\mu] = -\not p \gamma^\mu \gamma_\mu + 2p^\mu \gamma_\mu = -2\not p$

$Tr[\_\not p\not p] = p_\mu p_\nu Tr[\_\gamma^\mu\gamma^\nu] = -p_\mu p_\nu Tr[\_\gamma^\nu\gamma^\mu] + 2p^2 Tr[\_]$
$= -Tr[\_\not p\not p] + 2p^2 Tr[\_] = p^2 Tr[\_]$

✓

$$I_3 = \mathrm{Tr}\left\{(\not p' + m)\left[\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu\right](\not p + m)\left[\gamma_\mu \not k' \gamma_\nu - 2\gamma_\nu p_\mu\right]\right\}$$

Odd # vanishes

$$= \mathrm{Tr}\left\{\not p'\left[\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu\right]\not p\left[\gamma_\mu \not k' \gamma_\nu - 2\gamma_\nu p_\mu\right]\right\}$$

$$+ m^2\,\mathrm{Tr}\left\{\left[\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu\right]\left[\gamma_\mu \not k' \gamma_\nu - 2\gamma_\nu p_\mu\right]\right\}$$

$$= \mathrm{Tr}\Big\{\not p'\gamma^\mu \not k \underbrace{\gamma^\nu \not p \gamma_\mu \not k' \gamma_\nu}_{-2\not k'\gamma_\mu \not p} - 2\not p'\overbrace{\gamma^\mu \not k \underbrace{\gamma^\nu \not p \gamma_\nu}_{-2\not p} p_\mu}^{\not p}$$

$$+ 2\not p'\underbrace{\gamma^\mu p^\nu \not p \gamma_\mu}_{-2\not p} \not k' \underbrace{\gamma_\nu}_{\not p} - 4\not p'\underbrace{\gamma^\mu p^\nu \not p \gamma_\nu p_\mu}_{\not p\ \not p}\Big\}$$

$$+ m^2\,\mathrm{Tr}\Big\{\gamma^\mu \not k \underbrace{\gamma^\nu \gamma_\mu \not k' \gamma_\nu}_{4k'_\mu} + 2\underbrace{\gamma^\mu p^\nu \gamma_\mu}_{4} \not k' \underbrace{\gamma_\nu}_{\not p} - 2\underbrace{\gamma^\mu \not k \underbrace{\gamma^\nu \gamma_\nu}_{4} p_\mu}_{\not p} - 4\underbrace{\gamma^\mu \underbrace{p^\nu \gamma_\nu}_{\not p} p_\mu}_{\not p}\Big\}$$

$$= \mathrm{Tr}\Big\{-2\not p'\underbrace{\gamma^\mu \not k \not k' \gamma_\mu}_{4(k\cdot k')} \not p + 4\not p'\not p \not k \not p - 4\not p'\not p \not k' \not p - 4\not p'\not p \not p \not p\Big\}$$

$$+ m^2\,\mathrm{Tr}\left\{4\not k'\not k + 8\not k'\not p - 8\not p \not k - 4\not p \not p\right\}$$

$$= -8(k\cdot k')\,\mathrm{Tr}\{\not p'\not p\} - 4m^2\,\mathrm{Tr}\{\not p'\not p\}$$

$$+ 16\left\{(p'\cdot p)(k\cdot p) - (p'\cdot k)(p\cdot p) + (p'\cdot p)(k\cdot p)\right\}$$

$$- 16\left\{(p'\cdot p)(k'\cdot p) - (p'\cdot k')(p\cdot p) + (p'\cdot p)(p\cdot k')\right\}$$

$$+ 4m^2\left(4(k\cdot k')\right) + 8m^2\left(4(k'\cdot p)\right) - 8m^2\left(4(p\cdot k)\right) - 4m^2\left(4(p\cdot p)\right)$$

$$= -32(k\cdot k')(p'\cdot p) - 16m^2(p'\cdot p) + 32(p'\cdot p)(p\cdot k) - 32(p'\cdot p)(p\cdot k')$$

$$- 16m^2(p\cdot k) + 16m^2(p\cdot k) + 16m^2(k\cdot k') + 32m^2(p\cdot k')$$

$$- 32m^2(p\cdot k) - 16m^4$$

$k+p=k'+p'$

$$= -32m^2\left\{(p\cdot k) - (p\cdot k')\right\} + 16m^2\left\{(p\cdot k) - (p\cdot k')\right\} + 16m^2 k(\cancel{k} + p - p') - 16m^4$$

$$- 32(p'\cdot p)\,k(\cancel{k} + p - p') - 16m^2(p'\cdot p) + 32(p'\cdot p)(p\cdot k) - 32(p'\cdot p)(p\cdot k')$$

$$= -32m^2\left\{(p\cdot k) - (p\cdot k')\right\} + 32m^2\left\{(p\cdot k) - (p\cdot k')\right\} - 16m^4 - 16m^2(p'\cdot p)$$

$$= -16m^4 - 16m^2 (p' \cdot p) = -16m^4 - 16m^2 \left((p \cdot k) - (p \cdot k') + m^2\right)$$

$$= -32m^4 - 16m^2 \left\{ (p \cdot k) - (p \cdot k') \right\}$$

$$= -8 \left\{ 4m^4 + 2m^2 \left[(p \cdot k) - (p \cdot k') \right] \right\}$$

f)

Where from? $\frac{\omega'}{\omega}$ in $d\sigma/d\Omega$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{8\pi}\frac{1}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \overline{|M^2|}$$

$$= \frac{1}{8\pi}\frac{1}{4m^2}\left(\frac{\omega'}{\omega}\right)^2 \frac{e^4}{4}\left\{\frac{I_1}{(2(k\cdot p))^2} + \frac{I_2}{(2(k'\cdot p))^2} + \frac{I_3+I_4}{2(k\cdot p)(k'\cdot p)}\right\}$$

$\alpha = \frac{e^2}{4\pi}$

Why can we write k' like this in lab frame?

$$\overset{\downarrow}{=} \frac{16\pi^2\alpha^2}{8\pi\cdot 16m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{I_1}{(2(k\cdot p))^2} + \frac{I_2}{(2(k'\cdot p))^2} + \frac{I_3+I_4}{2(k\cdot p)(k'\cdot p)}\right\}$$

$$= \frac{\pi\alpha^2}{8m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{I_1}{(2(k\cdot p))^2} + \frac{I_2}{(2(k'\cdot p))^2} + \frac{I_3+I_4}{2(k\cdot p)(k'\cdot p)}\right\}$$

$$I_1 = 16\{2m^4 + 2m^2 p\cdot k + 2(p\cdot k')(p\cdot k)\}$$

$$= 16\{2m^4 + 2m^2(m\omega) + 2m^2\omega\omega'\} \quad \leftarrow (*)$$

$\forall\ p = \begin{pmatrix} m\\0\\0\\0\end{pmatrix},\ k = \begin{pmatrix}\omega\\0\\0\\\omega\end{pmatrix}$

$$I_2 = 16\{2m^4 - 2m^2(m\omega') + 2m^2\omega\omega'\}$$

$k' = \begin{pmatrix}\omega'\\ \omega'\sin\theta\\ 0\\ \omega'\cos\theta\end{pmatrix}$

$$I_3 = -8\{4m^4 + 2m^2(m\omega - m\omega')\}$$

$$= I_4$$

$$(p')^2 = m^2 = (k+p-k')^2 = m^2 + 2kp - 2k'p - 2kk'$$

$$\Leftrightarrow\quad kk' = kp - k'p$$

$$\overset{(*)}{\Leftrightarrow}\quad \omega\omega'(1-\cos\theta) = m\omega - m\omega' = m(\omega-\omega') \quad (**)$$

$$= \frac{\pi\alpha^2}{8m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{16(2m^4+2m^2(m\omega)+2m^2\omega\omega')}{4m^2\omega^2} + \frac{16(2m^4-2m^2(m\omega')+2m^2\omega\omega')}{4m^2\omega'^2} - \frac{16(4m^4+2m^2(m\omega-m\omega'))}{4m^2\omega\omega'}\right\}$$

$$= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{m^4\left(\frac{1}{m^2\omega^2}+\frac{1}{m^2\omega'^2}-\frac{2}{m^2\omega\omega'}\right) + m^2\left(\frac{m\omega}{m^2\omega^2}-\frac{m\omega'}{m^2\omega'^2}\right) + \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - m\frac{(\omega-\omega')}{\omega\omega'}\right\}$$

$$= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + m^2\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)^2 + m\left(\frac{1}{\omega}-\frac{1}{\omega'}\right) - m\frac{\omega-\omega'}{\omega\omega'}\right\}$$

$$\overset{(**)}{=} \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left\{\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + (1-\cos\theta)^2 - (1-\cos\theta) - (1-\cos\theta)\right\}$$

$$= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left\{ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 1 + \cos^2\theta - 2\cos\theta - 2 + 2\cos\theta \right\}$$

$$= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left\{ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - (1 - \cos^2\theta) \right\}$$

$$= \frac{\pi\alpha}{m^2}\left(\frac{\omega'}{\omega}\right)^2 \left\{ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta \right\}$$