Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

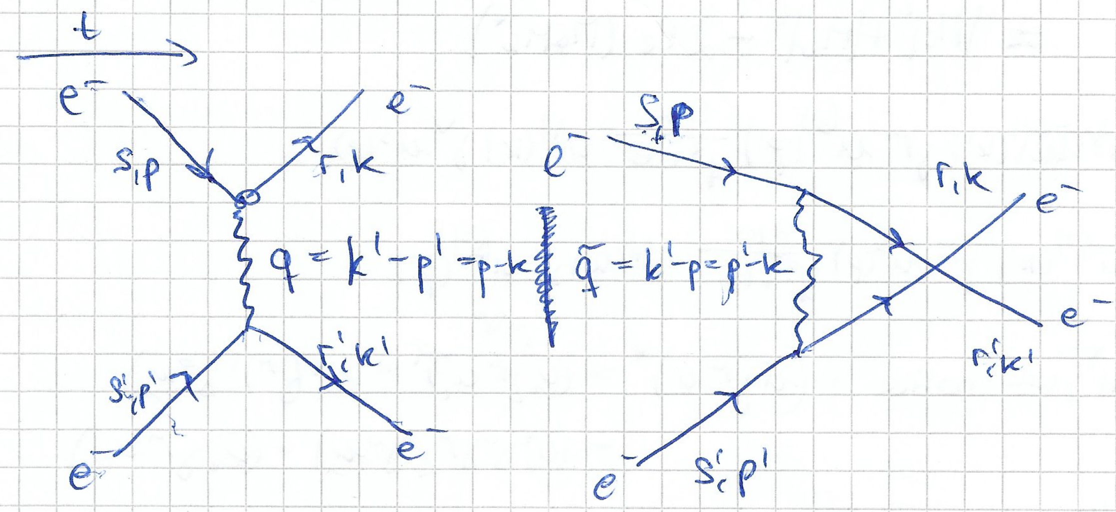
I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

19.07.2017 Quantum Field theory Exercise 11 Marvin Zeuka

H1.2)

$$e^{(s)}(p) + e^{(s')}(p') \longrightarrow e^{(r)}(k) + e^{(r')}(k')$$

$$p + p' = k + k'$$



t-channel $\hat{=} M_a$ u-channel $\hat{=} M_b$

$$iM_a = \bar{u}^{r'}(k')(-ie\gamma^\mu)u^{s'}(p')\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\bar{u}^r(k)(-ie\gamma^\nu)u^s(p)$$

$$iM_b = \bar{u}^{r'}(k')(-ie\gamma^\mu)u^{s}(p)\frac{-ig_{\mu\nu}}{\tilde{q}^2+i\epsilon}\bar{u}^r(k)(-ie\gamma^\nu)u^{s'}(p')$$

keine Symmetriefaktoren?

For the minus sign, we look at: $\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu$

$\rightsquigarrow \mathcal{H}_{int} = e\bar{\psi}\gamma^\mu\psi A_\mu$

$$\langle kk'|\underbrace{T\{1 - ie\int d^4x\,\bar{\psi}_x\gamma^\mu_x\psi_x A^x_\mu - \tfrac{1}{2}e^2\int d^4x\,d^4y\,\bar{\psi}_x\gamma^\mu_x\psi_x A^x_\mu\bar{\psi}_y\gamma^\nu_y\psi_y A^y_\nu + \dots}_{T\exp(-i\int d^4x\,\mathcal{H}_{int})}|pp'\rangle$$

Is this the normal ordered part of the time ordered product which contributes here?

2nd order diagrams: $-\frac{1}{2}e^2\int d^4x\,d^4y\,\langle kk'|:\bar{\psi}_x\gamma^\mu_x\psi_x A_\mu\bar{\psi}_y\gamma^\nu\psi_y A_\nu:|pp'\rangle$

$(|pk\rangle)^\dagger = (a_p^\dagger a_k^\dagger|0\rangle)^\dagger = \langle 0|a_k a_p = \langle pk|$

t-channel: $(+1)$ 2 swappings for fermions

$$\langle 0|a_{k'}a_k\bar{\psi}_x\gamma^\mu_x\psi_x A_\mu\bar{\psi}_y\gamma^\nu_y\psi_y A_\nu a_p^\dagger a_{p'}^\dagger|0\rangle$$

What happens if we swap photon field with fermion field?

u-channel (-1) 1 swapping

$\rightsquigarrow$ relative minus sign between those two!

$\{a_k, a_k^\dagger\} \hat{=}$ propagator

was wenn $\{a_k, a_{k'}\} \to (-1)$?

building to $|pp'\rangle$ oder $|p'p\rangle$ hier?

b) $M = M_a - M_b$, where the $(-)$ accounts for the relative minus sign in our definition for M_a, M_b

$$\Rightarrow |M|^2 = |M_a - M_b|^2 = (M_a - M_b)(M_a^* - M_b^*) = |M_a|^2 + |M_b|^2 - M_b M_a^* - M_a M_b^*$$
$$= |M_a|^2 + |M_b|^2 - 2\,\mathrm{Re}\,(M_a M_b^*)$$

$M_a M_b^* = M_b^* M_a$

$$M_a = e^2\, \bar{u}^{(r')}(k')\gamma^\mu u^{(s')}(p')\frac{g_{\mu\nu}}{q^2+i\epsilon}\bar{u}^{(r)}(k)\gamma^\nu u^{s}(p)$$

$$M_b = \bar{u}^{(r')}(k')\gamma^\mu u^{(s)}(p)\frac{g_{\mu\nu}}{q^2+i\epsilon}\bar{u}^{(r)}(k)\gamma^\nu u^{(s')}(p')$$

$i\epsilon$ im Nenner weg?

For M_i^*, we notice, $(\bar{u}\gamma^\mu u)^* = (\bar{u}_2\gamma^\mu u_1)^\dagger = (u_1^\dagger(\gamma^\mu)^\dagger\gamma^0 u_2)$
$$= (u_1^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0 u_2) = (\bar{u}_1\gamma^\mu u_2)$$

$$|M_a|^2 = \frac{e^4}{q^4}\bar{u}^{r'}(k')\gamma^\mu u^{s'}(p')g_{\mu\nu}\bar{u}^r(k)\gamma^\nu u^s(p)\,\bar{u}^{s'}(p')\gamma^\sigma u^{r'}(k')g_{\sigma\lambda}\bar{u}^s(p)\gamma^\lambda u^r(k)$$

2 verschiedene $\mu\nu$, (ρ,ν)?

scalars
$$= \frac{e^4}{q^4}\left[\bar{u}^{r'}(k')\gamma^\mu u^{s'}(p')\bar{u}^{s'}(p')\gamma^\sigma u^{r'}(k')\right]\left[\bar{u}^r(k)\gamma_\mu u^s(p)\bar{u}^s(p)\gamma_\sigma u^r(k)\right]$$

With $\overline{|M|^2} = \frac{1}{4}\sum_{s,s',r,r'}|M|^2 = \frac{1}{4}\sum_{s,r,s',r'}|M_a|^2 + |M_b|^2 - 2\,\mathrm{Re}\,(M_a M_b^*)$
$$= \overline{|M_a|^2} + \overline{|M_b|^2} - \overline{2\,\mathrm{Re}(M_a M_b^*)}$$

We find:

$$\overline{|M_a|^2} = \frac{e^4}{4q^4}\sum_{s,s',r,r'}\mathrm{Tr}\left\{\bar{u}^{r'}(k')\gamma^\mu u^{s'}(p')\bar{u}^{s'}(p')\gamma^\sigma u^{r'}(k')\right\}\mathrm{Tr}\left\{\bar{u}^r(k)\gamma_\mu u^s(p)\bar{u}^s(p)\gamma_\sigma u^r(k)\right\}$$

$\sum_s u_p^s \bar{u}_p^s = \not{p}+m$
$$\Rightarrow \frac{e^4}{4q^4}\underbrace{\mathrm{Tr}\left\{\gamma^\mu(\not{p}'+m)\gamma^\sigma(\not{k}'+m)\right\}}_{(1)}\underbrace{\mathrm{Tr}\left\{\gamma_\mu(\not{p}+m)\gamma_\sigma(\not{k}+m)\right\}}_{\text{analogue to }(1)}$$

odd # vanishes

$$(1) = \mathrm{Tr}\left\{\gamma^\mu p'_\kappa\gamma^\kappa\gamma^\sigma k'_\lambda\gamma^\lambda + m^2\gamma^\mu\gamma^\sigma\right\}$$
$$= p'_\kappa k'_\lambda\,\mathrm{Tr}\{\gamma^\mu\gamma^\kappa\gamma^\sigma\gamma^\lambda\} + m^2\,\mathrm{Tr}\{\gamma^\mu\gamma^\sigma\}$$
$$= 4p'_\kappa k'_\lambda\left\{g^{\mu\kappa}g^{\sigma\lambda} - g^{\mu\sigma}g^{\kappa\lambda} + g^{\mu\lambda}g^{\kappa\sigma}\right\} + 4m^2 g^{\mu\sigma}$$
$$= 4p'^\mu k'^\sigma - 4g^{\mu\sigma}(p'\cdot k') + 4p'^\sigma k'^\mu + 4m^2 g^{\mu\sigma}$$

not really $m^2 = 0$ but small to other energies?

$$= \frac{e^4}{4q^4}16\left\{p'^\mu k'^\sigma - g^{\mu\sigma}(p'\cdot k') + p'^\sigma k'^\mu + \cancel{m^2 g^{\mu\sigma}}\right\}\left\{p_\mu k_\sigma - g_{\mu\sigma}(p\cdot k) + p_\sigma k_\mu + \cancel{m^2 g_{\mu\sigma}}\right\}$$

$$= \frac{4e^4}{q^4}\Big\{(p\cdot p')(k\cdot k') - (p'\cdot k')(p\cdot k) + (p'\cdot k)(p\cdot k')$$
$$- (p\cdot k)(p'\cdot k') + 4(p\cdot k)(p'\cdot k') - (p\cdot k)(p'\cdot k')$$
$$+ (p'\cdot k)(p\cdot k') - (p'\cdot k')(p\cdot k) + (p\cdot p')(k\cdot k')\Big\}$$

$$= \frac{4e^4}{q^4}\Big\{2(p\cdot p')(k\cdot k') + 2(p'\cdot k)(p\cdot k')\Big\}$$

$$= \frac{8e^4}{q^4}\Big\{(k\cdot k')(p\cdot p') + (p\cdot k')(k\cdot p')\Big\}, \quad q = p-k = k'-p'$$

(see 1^{st} page)

change nothing but $k \leftrightarrow k'$?

For $|M_b|^2$, we notice that the diagrams for M_a and M_b just have $k \leftrightarrow k'$ exchanged and thus will yield the same result with $k \leftrightarrow k'$ interchanged: ✓

$$|M_b|^2 = \frac{8e^4}{(p-k')^4}\Big\{(k\cdot k')(p\cdot p') + (p\cdot k)(k'\cdot p')\Big\}$$

$$= \frac{8e^4}{\tilde{q}^4}\Big\{(k\cdot k')(p\cdot p') + (p\cdot k)(k'\cdot p')\Big\}, \quad \tilde{q} = p-k' = k-p'$$

(see 1^{st} page)

Here we have used: $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\{\gamma^\mu,\gamma^\nu\} - \gamma^\nu\gamma^\mu)$
$$= \mathrm{Tr}(2g^{\mu\nu}\mathbb{1}) - \mathrm{Tr}(\gamma^\mu\gamma^\nu)$$
$$\Leftrightarrow \mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\kappa\gamma^\sigma\gamma^\lambda) = 4(g^{\mu\kappa}g^{\sigma\lambda} - g^{\mu\sigma}g^{\kappa\lambda} + g^{\mu\lambda}g^{\kappa\sigma})$$

(see last sheet)

$$\overline{-2\,\mathrm{Re}(M_a M_b^*)} = \frac{-2e^4}{4q^2\tilde{q}^2}\sum_{s,s',r,r'}\mathrm{Re}\Big\{\bar{u}^{r'}(k')\gamma^\mu u^{s'}(p')g_{\mu\nu}\bar{u}^r(k)\gamma^\nu u^s(p)$$
$$\times\, \bar{u}^s(p)\gamma^\sigma u^{r'}(k')g_{\sigma\lambda}\bar{u}^{s'}(p')\gamma^\lambda u^r(k)\Big\}$$

$$= \frac{-2e^4}{2q^2\tilde{q}^2}\mathrm{Re}\sum_{s,s',r,r'}\mathrm{Tr}\Big\{\bar{u}^{s'}(p')\gamma^\lambda u^r(k)\bar{u}^r(k)\gamma^\nu u^s(p)\bar{u}^s(p)\gamma_\lambda u^{r'}(k')\bar{u}^{r'}(k')\gamma_\nu u^{s'}(p')\Big\}$$

$$= -\frac{2e^4}{2q^2\tilde{q}^2}\mathrm{Re}\,\mathrm{Tr}\Big\{\gamma^\lambda(\not{k}+m)\gamma^\nu(\not{p}+m)\gamma_\lambda(\not{k}'+m)\gamma_\nu(\not{p}'+m)\Big\}$$

odd # vanishes

$$= -\frac{2e^4}{2q^2\tilde{q}^2}\mathrm{Re}\,\mathrm{Tr}\Big\{\gamma^\lambda k_\mu\gamma^\mu\gamma^\nu p_\sigma\gamma^\sigma\gamma_\lambda k'_\delta\gamma^\delta\gamma_\nu p'_\epsilon\gamma^\epsilon + \underbrace{m^2\{\dots\}}_{=0}$$
$$+ \underbrace{m^4\{\dots\}}_{=0}\Big\}$$

$$= -\frac{2e^4}{2q^2\tilde{q}^2} \mathrm{Re}\{\mathrm{Tr}\{\gamma^\lambda k_\mu \gamma^\mu \gamma^\nu p_\sigma \gamma^\sigma \gamma_\lambda k'_\delta \gamma^\delta \gamma_\nu p'_\epsilon \gamma^\epsilon\}\}$$

Wrong hints given?!

$$= -\frac{2e^4}{2q^2\tilde{q}^2} \mathrm{Re}\{k_\mu p_\sigma k'_\delta p'_\epsilon \mathrm{Tr}\{\underbrace{\gamma^\lambda \gamma^\mu \gamma^\nu \gamma^\sigma \gamma_\lambda}_{-2\gamma^\sigma \gamma^\nu \gamma^\mu} \gamma^\delta \gamma_\nu \gamma^\epsilon\}\}$$

$$-2\gamma^\sigma \underbrace{\gamma^\nu \gamma^\mu \gamma^\delta \gamma_\nu}_{4g^{\mu\delta}}$$

$$= \frac{4e^4}{q^2\tilde{q}^2} \mathrm{Re}\{k_\mu p_\sigma k'_\delta p'_\epsilon \mathrm{Tr}\{\gamma^\sigma g^{\mu\delta} \gamma^\epsilon\}\}$$

$$= \frac{4e^4}{q^2\tilde{q}^2} \mathrm{Re}\{(k\cdot k') p_\sigma p'_\epsilon (4g^{\sigma\epsilon})\} = \frac{16e^4}{q^2\tilde{q}^2}\{(k\cdot k')(p\cdot p')\}$$

Where we have used, $\gamma^\nu\gamma^\mu\gamma^\delta\gamma_\nu = -\gamma^\mu\gamma^\nu\gamma^\delta\gamma_\nu + 2g^{\nu\mu}\gamma^\delta\gamma_\nu$

$$= \gamma^\mu\gamma^\delta\gamma^\nu\gamma_\nu - 2\gamma^\mu g^{\nu\delta}\gamma_\nu + 2g^{\nu\mu}\gamma^\delta\gamma_\nu$$

$$= 4\gamma^\mu\gamma^\delta - 2\gamma^\mu\gamma^\delta + 2\gamma^\delta\gamma^\mu = 2\gamma^\mu\gamma^\delta + 2\gamma^\delta\gamma^\mu$$

$$= 2\{\gamma^\mu, \gamma^\delta\} = 4g^{\mu\delta}$$

and

$$\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\sigma\gamma_\lambda = -\gamma^\mu \underbrace{\gamma^\lambda\gamma^\nu\gamma^\sigma\gamma_\lambda}_{4g^{\nu\sigma}} + 2g^{\mu\lambda}\gamma^\nu\gamma^\sigma\gamma_\lambda$$

$$= -4g^{\nu\sigma}\gamma^\mu + 2\gamma^\nu\gamma^\sigma\gamma^\mu = -4g^{\nu\sigma}\gamma^\mu - 2\gamma^\sigma\gamma^\nu\gamma^\mu + 4g^{\nu\sigma}\gamma^\mu$$

$$= -2\gamma^\sigma\gamma^\nu\gamma^\mu$$

Last but not least we used: $\gamma_\mu\gamma^\mu = g_{\mu\nu}\gamma^\nu\gamma^\mu = \frac{1}{2}(g_{\mu\nu}\gamma^\nu\gamma^\mu + g_{\nu\mu}\gamma^\mu\gamma^\nu)$

$$= \frac{1}{2}g_{\mu\nu}\{\gamma^\nu, \gamma^\mu\} = g_{\mu\nu}g^{\nu\mu} = 4\cdot(\mathbb{1})$$

c) From now on, we want to be working in the CMS, where

$$p = \begin{pmatrix} E_1 \\ \vec{p} \end{pmatrix}, \quad p' = \begin{pmatrix} E_2 \\ -\vec{p} \end{pmatrix}, \quad k = \begin{pmatrix} E_3 \\ \vec{k} \end{pmatrix}, \quad k' = \begin{pmatrix} E_4 \\ -\vec{k} \end{pmatrix}$$

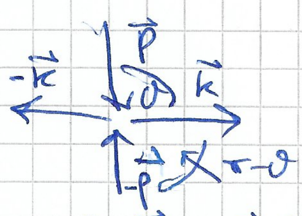
As we will be working out the case for highly relativistic electrons, we get $p^2 = 0 = E_1^2 - |\vec{p}|^2 \Leftrightarrow E_1 = |\vec{p}_1|$ and analogue for the other four-vectors. Thus

Wo genau ging dies im Lab frame schief? Also warum CMS und nicht wie letzte Aufgabe?

$$p = \begin{pmatrix} |\vec{p}| \\ \vec{p} \end{pmatrix}, \quad p' = \begin{pmatrix} |\vec{p}| \\ -\vec{p} \end{pmatrix}, \quad k = \begin{pmatrix} |\vec{k}| \\ \vec{k} \end{pmatrix}, \quad k' = \begin{pmatrix} |\vec{k}| \\ -\vec{k} \end{pmatrix}$$

$$\Rightarrow \quad p \cdot k = |\vec{p}||\vec{k}| - \vec{p} \cdot \vec{k}$$
$$= |\vec{p}||\vec{k}|\,(1 + \cos\vartheta)$$
$$= 2|\vec{p}||\vec{k}|\,\cos^2 \vartheta/2$$

Gilt im t-channel und $E_1 = E_{CMS}/2 = \sqrt{s}/2$?



$$\vec{p} \cdot \vec{k} = |\vec{p}||\vec{k}| \cos(\pi - \vartheta)$$
$$= -|\vec{p}||\vec{k}| \cos\vartheta$$

$$p \cdot k' = |\vec{p}||\vec{k}|\,(1 - \cos\vartheta)$$
$$= 2|\vec{p}||\vec{k}|\,\sin^2 \vartheta/2$$

$$p \cdot p' = 2|\vec{p}|^2$$
$$p' \cdot k = |\vec{p}||\vec{k}|\,(1 - \cos\vartheta) = 2|\vec{p}||\vec{k}| \sin^2 \vartheta/2$$
$$p' \cdot k' = |\vec{p}||\vec{k}|\,(1 + \cos\vartheta) = 2|\vec{p}||\vec{k}| \cos^2 \vartheta/2$$
$$k \cdot k' = 2|\vec{k}|^2$$

Hence

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|M|^2} = \frac{8e^4}{64\pi^2 s}\left\{ \frac{(k\cdot k')(p\cdot p') + (p\cdot k')(k\cdot p')}{(p-k)^4} + \frac{(k\cdot k')(p\cdot p') + (p\cdot k)(k'\cdot p')}{(p-k')^4} + \frac{2(k\cdot k')(p\cdot p')}{(p-k)^2(p-k')^2} \right\}$$

$$= \frac{e^4}{8\pi^2 s}\left\{ \frac{2|\vec{k}|^2\, 2|\vec{p}|^2 + 4|\vec{p}|^2|\vec{k}|^2 \sin^4 \vartheta/2}{(2pk)(2pk)} + \frac{4|\vec{k}|^2|\vec{p}|^2 + 4|\vec{p}|^2|\vec{k}|^2 \cos^4 \vartheta/2}{(2pk')(2pk')} + \frac{2 \cdot 4|\vec{k}|^2|\vec{p}|^2}{(2pk)(2pk')} \right\}$$

$$= \frac{e^4}{8\pi^2 s} \left\{ \frac{|\vec{k}|^2 |\vec{p}|^2 (1 + \sin^4 \theta/2)}{4|\vec{p}|^2 |\vec{k}|^2 \cos^4 \theta/2} + \frac{|\vec{k}|^2 |\vec{p}|^2 (1 + \cos^4 \theta/2)}{4|\vec{p}|^2 |\vec{k}|^2 \sin^4 \theta/2} + \frac{2|\vec{k}|^2 |\vec{p}|^2}{4|\vec{p}|^2 |\vec{k}|^2 \sin^2 \theta/2 \cos^2 \theta/2} \right\}$$

$$= \frac{e^4}{32\pi^2 s} \left\{ \frac{1 + \sin^4 \theta/2}{\cos^4 \theta/2} + \frac{1 + \cos^4 \theta/2}{\sin^4 \theta/2} + \frac{2}{\sin^2 \theta/2 \cos^2 \theta/2} \right\}$$

$s = 4E^2$ with E: energy of one of the electrons in CMS $\Rightarrow E = |\vec{p}|$ in CMS