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Quantum Field theory 2nd Exercise Sheet — Marvin Zanke

04.05.2017

H2) $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$

"Scalar" field theory? Alternative? ⇝ Spinor field theory (doesn't behave like Lorentz vector! no vector field!)

$$\phi(x) = \int \frac{d^3k}{\sqrt{2\omega_{\vec k}}(2\pi)^3}\left(e^{-ikx} b_{\vec k} + e^{ikx} c_{\vec k}^\dagger\right)$$

$e^{-i(k^0x^0 - \vec k \cdot \vec x)}$ $\qquad$ $e^{i(k^0x^0 - \vec k \cdot \vec x)}$

Why ϕ^* and not $\phi^\dagger$? ↓ is the same in this case

$$\phi^*(x) = \int \frac{d^3k}{\sqrt{2\omega_{\vec k}}(2\pi)^3}\left(e^{-ikx} c_{\vec k} + e^{ikx} b_{\vec k}^\dagger\right)$$

fulfilling the canonical commutation relations:

$$[b_{\vec k}, b_{\vec q}^\dagger] = [c_{\vec k}, c_{\vec q}^\dagger] = (2\pi)^3 \delta(\vec k - \vec q)$$

all others vanish

$$Q = \int d^3x\, i\{(\partial^0\phi^*)\phi - \phi^*(\partial^0\phi)\}$$

a) Using Heisenberg e.o.m.:

$$i\partial^0\phi = [\phi, H] = \phi H - H\phi \quad ⇝ -i\partial^0\phi^* = [\phi^*, H^*]$$

$$\Leftrightarrow -i\partial^0\phi^\dagger = H\phi^\dagger - \phi^\dagger H \Leftrightarrow i\partial^0\phi^* = [\phi^*, H]$$

Important here, whether $()^*$ or $()^\dagger$, because of $(AB)^* = A^*B^*$, $(AB)^\dagger = B^\dagger A^\dagger$? ⇝ need $()^\dagger$, as $H^\dagger \neq H^*$

and K.G. eq.:

$$(\partial^2 + m^2)\phi = 0 \Leftrightarrow \partial^{0\,2}\phi = \nabla^2\phi - m^2\phi$$

$$\Leftrightarrow \partial^{0\,2}\phi^* = \nabla^2\phi^* - m^2\phi^*$$

we find:

$$[H, Q] = \left[H, \int d^3x\, i\{(\partial^0\phi^*)\phi - \phi^*(\partial^0\phi)\}\right]$$

$$= \int d^3x\, [H, i\{(\partial^0\phi^*)\phi - \phi^*(\partial^0\phi)\}]$$

$$= \int d^3x\, \{i(\partial^0\phi^*)[H,\phi] + i[H,(\partial^0\phi^*)]\phi - i\phi^*[H,(\partial^0\phi)] - i[H,\phi^*](\partial^0\phi)\}$$

Heisenberg e.o.m.

$$= \int d^3x\, \{\cancel{(\partial^0\phi^*)(\partial^0\phi)} + i\{[H,(\partial^0\phi^*)]\phi - \phi^*[H,(\partial^0\phi)]\} - \cancel{(\partial^0\phi^*)(\partial^0\phi)}\}$$

again, Heisenberg e.o.m. implies $\partial^0 H = 0$

$$= \int d^3x\, i\{\partial^0[H,\phi^*]\phi - \phi^*\partial^0[H,\phi]\}$$

Heisenberg

$$= \int d^3x\, i\{\partial^0(-i\partial^0\phi^*)\phi - \phi^*\partial^0(-i\partial^0\phi)\}$$

think about it :-)

$$= \int d^3x\, \{(\partial^{0\,2}\phi^*)\phi - \phi^*(\partial^{0\,2}\phi)\}$$

K.G. eq.

$$= \int d^3x\, \{((\nabla^2 - m^2)\phi^*)\phi - \phi^*((\nabla^2 - m^2)\phi)\}$$

Do ϕ^* and ϕ still commute now? ⇝ quantized

$$= \int d^3x \left\{ (\nabla^2\phi^*)\phi - \phi^*(\nabla^2\phi) \right\} \quad (*)$$

Two ways to see that $(*)$ vanishes, first uses that $\phi(x) \to 0$, which might not be satisfied for these operators:

(1) $$(*) = \int d^3x\, \nabla \underbrace{\left((\nabla\phi^*)\phi - \phi^*(\nabla\phi) \right)}_{=: \vec{J}(\phi)} = \int d\vec{S} \cdot \vec{J}(\phi) \overset{\text{neglect surface terms}}{=} 0$$

V

(2) $$\nabla^2\phi^* = \int \frac{d^3k}{\sqrt{2\omega_{\vec{k}}(2\pi)^3}} \left(-|\vec{k}|^2 e^{-ikx} c_{\vec{k}} - |\vec{k}|^2 e^{ikx} b_{\vec{k}}^\dagger \right)$$

$$\overset{k^2 = |\vec{k}|^2}{=} - \int \frac{d^3k\, k^2}{\sqrt{2\omega_{\vec{k}}}(2\pi)^3} \left(e^{-ikx} c_{\vec{k}} + e^{ikx} b_{\vec{k}}^\dagger \right)$$

$$\nabla^2\phi = - \int \frac{d^3k\, k^2}{\sqrt{2\omega_{\vec{k}}}(2\pi)^3} \left(e^{-ikx} b_{\vec{k}} + e^{ikx} c_{\vec{k}}^\dagger \right)$$

$$\leadsto (*) = \int d^3x \left\{ - \int \frac{d^3k\, k^2}{\sqrt{2\omega_{\vec{k}}}(2\pi)^3} \left(e^{-ikx} c_{\vec{k}} + e^{ikx} b_{\vec{k}}^\dagger \right) \int \frac{d^3k'}{\sqrt{2\omega_{\vec{k}'}}(2\pi)^3} \left(e^{-ik'x} b_{\vec{k}'} + e^{ik'x} c_{\vec{k}'}^\dagger \right) \right.$$
$$\left. + \int \frac{d^3k}{\sqrt{2\omega_{\vec{k}}}(2\pi)^3} \left(e^{-ikx} c_{\vec{k}} + e^{ikx} b_{\vec{k}}^\dagger \right) \int \frac{d^3k'\, k'^2}{\sqrt{2\omega_{\vec{k}'}}(2\pi)^3} \left(e^{-ik'x} b_{\vec{k}'} + e^{ik'x} c_{\vec{k}'}^\dagger \right) \right\}$$

$$= \int d^3x \left\{ \int \frac{d^3k\, d^3k'\, k'^2}{2\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}}(2\pi)^6} \left(e^{-ikx} c_{\vec{k}} + e^{ikx} b_{\vec{k}}^\dagger \right) \left(e^{-ik'x} b_{\vec{k}'} + e^{ik'x} c_{\vec{k}'}^\dagger \right) \right.$$
$$\left. - \int \frac{d^3k\, d^3k'\, k^2}{2\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}}(2\pi)^6} \left(e^{-ikx} c_{\vec{k}} + e^{ikx} b_{\vec{k}}^\dagger \right) \left(e^{-ik'x} b_{\vec{k}'} + e^{ik'x} c_{\vec{k}'}^\dagger \right) \right\}$$

$$\frac{1}{(2\pi)^3} \int e^{ikx} d^3x = \delta(k)$$

$$\omega_{\vec{k}}^2 = \vec{k}^2 + m^2$$
$$\omega_{\vec{k}}^2 = \omega_{-\vec{k}}^2$$

$$= \int \frac{d^3k\, k^2}{2\omega_{\vec{k}}(2\pi)^6} \left\{ c_{\vec{k}} b_{-\vec{k}} + c_{\vec{k}} c_{\vec{k}}^\dagger + b_{\vec{k}}^\dagger b_{\vec{k}} + b_{\vec{k}}^\dagger c_{-\vec{k}}^\dagger \right\}$$
$$- \int \frac{d^3k\, k^2}{2\omega_{\vec{k}}(2\pi)^6} \left\{ c_{\vec{k}} b_{\vec{k}} + c_{\vec{k}} c_{\vec{k}}^\dagger + b_{\vec{k}}^\dagger b_{\vec{k}} + b_{\vec{k}}^\dagger c_{\vec{k}}^\dagger \right\}$$

$$= 0$$

b) We will need:

$$\partial^0 \phi = \int \frac{d^3k}{\sqrt{2\omega_{\vec k}}(2\pi)^3} (-ik^0) \left(e^{-ikx} b_{\vec k} - e^{ikx} c_{\vec k}^{+}\right)$$

$$\partial^0 \phi^* = \int \frac{d^3k}{\sqrt{2\omega_{\vec k}}(2\pi)^3} (-ik^0) \left(e^{-ikx} c_{\vec k} - e^{ikx} b_{\vec k}^{+}\right)$$

$$\leadsto Q = \int d^3x\, i \left\{ (\partial^0 \phi^*)\phi - \phi^*(\partial^0 \phi) \right\}$$

$$= \int d^3x\, i \left\{ \int \frac{d^3k\, d^3k'\, (-ik^0)}{2\sqrt{\omega_{\vec k}\omega_{\vec k'}}(2\pi)^6} \left(e^{-ikx} c_{\vec k} - e^{ikx} b_{\vec k}^{+}\right)\left(e^{-ik'x} b_{\vec k'} + e^{ik'x} c_{\vec k'}^{+}\right) - \int \frac{d^3k\, d^3k'\, (-ik^0)}{2\sqrt{\omega_{\vec k}\omega_{\vec k'}}(2\pi)^6} \left(e^{-ik'x} c_{\vec k'} + e^{ik'x} b_{\vec k'}^{+}\right)\left(e^{-ikx} b_{\vec k} - e^{ikx} c_{\vec k}^{+}\right) \right\}$$

$$= \int \frac{d^3x\, d^3k\, d^3k'}{2\sqrt{\omega_{\vec k}\omega_{\vec k'}}(2\pi)^6} k^0 \left\{ \left(e^{-ikx} c_{\vec k} - e^{ikx} b_{\vec k}^{+}\right)\left(e^{-ik'x} b_{\vec k'} + e^{ik'x} c_{\vec k'}^{+}\right) - \left(e^{-ik'x} c_{\vec k'} + e^{ik'x} b_{\vec k'}^{+}\right)\left(e^{-ikx} b_{\vec k} - e^{ikx} c_{\vec k}^{+}\right) \right\}$$

$e^{ikx} = e^{i(k^0x^0 - \vec k \vec x)}$

$e^{-ikx} = e^{-i(k^0x^0 - \vec k \vec x)}$

$\int d^3x\, e^{i\vec k \vec x} = (2\pi)^3 \delta(\vec k)$ where the $e^{\pm ik^0x^0}$ remains

can choose $x^0 = t = 0$

$$\overset{\downarrow}{=} \int \frac{d^3k\, d^3k'\, k^0}{2\sqrt{\omega_{\vec k}\omega_{\vec k'}}(2\pi)^3} \Big\{ e^{-ix^0(k^0+k'^0)} \delta(k+k') c_{\vec k} b_{\vec k'} + e^{ix^0(k'^0-k^0)} \delta(k-k') c_{\vec k} c_{\vec k'}^{+}$$
$$- e^{ix^0(k^0-k'^0)} \delta(k'-k) b_{\vec k}^{+} b_{\vec k'} - e^{ix^0(k'^0+k^0)} \delta(k'+k) b_{\vec k}^{+} c_{\vec k'}^{+}$$
$$- e^{-ix^0(k'^0+k^0)} \delta(k'+k) c_{\vec k'} b_{\vec k} + e^{ix^0(k^0-k'^0)} \delta(k'-k) c_{\vec k'} c_{\vec k}^{+}$$
$$- e^{ix^0(k'^0-k^0)} \delta(k-k') b_{\vec k'}^{+} b_{\vec k} + e^{ix^0(k'^0+k^0)} \delta(k'+k) b_{\vec k'}^{+} c_{\vec k}^{+} \Big\}$$

$\omega_{\vec k'}^2 = \omega_{-\vec k}^2 = \omega_{\vec k}^2$

$k^{0\,2} = \omega_{\vec k}^2$

$$\overset{\downarrow}{=} \int \frac{d^3k}{2\cdot(2\pi)^3} \Big\{ \underline{e^{-ix^0(k^0+k'^0)} c_{\vec k} b_{-\vec k}} + e^{ix^0(k'^0-k^0)} c_{\vec k} c_{\vec k}^{+} - e^{ix^0(k^0-k'^0)} b_{\vec k}^{+} b_{\vec k}$$
$$\underline{- e^{ix^0(k'^0+k^0)} b_{\vec k}^{+} c_{-\vec k}^{+}} \underline{- e^{-ix^0(k'^0+k^0)} c_{-\vec k} b_{\vec k}} + e^{ix^0(k^0-k'^0)} c_{\vec k} c_{\vec k}^{+}$$
$$- e^{ix^0(k'^0-k^0)} b_{\vec k}^{+} b_{\vec k} \underline{+ e^{ix^0(k'^0+k^0)} b_{\vec k}^{+} c_{\vec k}^{+}} \Big\}$$

the simply underlined parts vanish if we use $\vec k' \to -\vec k$ for half of the terms

$$\overset{\downarrow}{=} \int \frac{d^3k}{2\cdot(2\pi)^3} \Big\{ e^{ix^0(k'^0-k^0)} \Big(\overbrace{[c_{\vec k}, c_{\vec k}^{+}]}^{\propto\, \delta^{(3)}(0)} + c_{\vec k}^{+} c_{\vec k} \Big) + e^{ix^0(k^0-k'^0)} \Big(\overbrace{[c_{\vec k}, c_{\vec k}^{+}]}^{\propto\, \delta^{(3)}(0)} + c_{\vec k}^{+} c_{\vec k} \Big)$$
$$- e^{ix^0(k^0-k'^0)} b_{\vec k}^{+} b_{\vec k} - e^{ix^0(k'^0-k^0)} b_{\vec k}^{+} b_{\vec k} \Big\}$$

$[H, Q] = 0$

$\leadsto \partial^0 Q = 0$

$\leadsto$ evaluate at $t=0$

$$\overset{\downarrow}{=} \int \frac{d^3k}{(2\pi)^3} \left(c_{\vec k}^{+} c_{\vec k} - b_{\vec k}^{+} b_{\vec k} \right) ✓$$