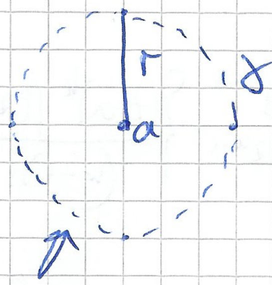
Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

12.05.2017 Quantum Field Theory 3rd Exercise Sheet Marvin Zanke

H3) a) $f_n(z) = (z-a)^n \quad , n \in \mathbb{Z}$



$$\int_\gamma dz\, f_n(z) = \int_\gamma dz\, (z-a)^n$$

$$\overset{(*)}{=} \int_0^1 dt\, (2\pi i r e^{2\pi i t})(r e^{2\pi i t})^n$$

$$= 2\pi i \int_0^1 dt\, (r e^{2\pi i t})^{n+1}$$

(*) Parametrization: $\gamma: z(t) = r e^{2\pi i t} + a$

$$\frac{dz}{dt} = 2\pi i r e^{2\pi i t}$$

(1) $= 2\pi i \int_0^1 dt\, 1 = 2\pi i$ for $n = -1$ ✓

(2) $= 2\pi i \int_0^1 dt\, r^{n+1} e^{2\pi i (n+1) t}$

$$= 2\pi i r^{n+1} \left[\frac{1}{2\pi i (n+1)} e^{2\pi i (n+1) t}\right]\Bigg|_0^1 = \frac{r^{n+1}}{n+1} \underbrace{\left(e^{2\pi i (n+1)} - e^{0}\right)}_{=0}$$

$= 0$ for $n \neq -1$ ✓

+

b) $f(z) = \sum_{n=-k}^{\infty} b_n (z-a)^n$

Kann man alles in Laurent-Reihe entwickeln oder gewisse Voraussetzungen wie bei Taylorreihe dass keine Singularitäten?

$$\oint_\gamma f(z)\, dz = \oint_\gamma \sum_{n=-k}^{\infty} b_n (z-a)^n = \sum_{n=-k}^{\infty} b_n \oint_\gamma (z-a)^n dz$$

$$= \sum_{n=-k}^{\infty} b_n \left(2\pi i\, \delta_{n,-1}\right) = 2\pi i\, b_{-1} =: 2\pi i \operatorname{Res}(f,a)$$

✓ +

c) $f(z) = \sum_{n=-k}^{\infty} b_n (z-a)^n = b_{-k}(z-a)^{-k} + b_{-k+1}(z-a)^{-k+1} + \dots$
$+ b_0 (z-a)^0 + b_1 (z-a)^1 + \dots$

$$\Leftrightarrow (z-a)^k f(z) = b_{-k} + b_{-k+1}(z-a) + \dots + b_0 (z-a)^k + b_1 (z-a)^{k+1} + \dots$$

$$\Rightarrow \frac{d}{dz}\left\{(z-a)^k f(z)\right\} = b_{-k+1} + b_{-k+2}\cdot 2(z-a) + \dots + b_0 \cdot k (z-a)^{k-1} + b_1 \cdot (k+1)(z-a)^k + \dots$$

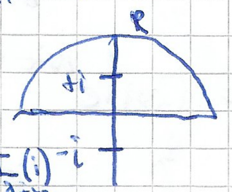
$$\Rightarrow \frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\} = \underbrace{b_{-k+(k-1)}} \cdot (k-1)! + b_0 \cdot k!(z-a) + b_1 (k+1)!(z-a)^2 + \dots$$

$$\Rightarrow \frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\}\Big|_{z=a} = \left\{b_{-1}\cdot(k-1)! + b_0\, k!\,(z-a) + b_1\,(k+1)!\,(z-a)^2 + \dots\right\}\Big|_{z=a}$$

$$= b_{-1}\cdot(k-1)!$$

$$\Leftrightarrow \frac{d^{k-1}}{dz^{k-1}}\left\{\frac{(z-a)^k f(z)}{(k-1)!}\right\} = b_{-1} \equiv \mathrm{Res}(f,a)$$ ✓ ⊕

Simply connected open subsets? What for?



d) $$\int_{-\infty}^{\infty} \underbrace{\frac{e^{ix}}{x^2+1}}_{=:g(x)} dx \overset{\text{see later}}{=} \int_{\gamma_1+\gamma_2} \frac{e^{ix}}{x^2+1}\,dx = 2\pi i\,\mathrm{Res}(g,i)\, I_{\gamma_1+\gamma_2}(i)$$

$(x^2+1) = (x+i)(x-i)$, but only $(x-i)$ in upper arc

winding number

$$\overset{\text{positive; c) with } k=1}{=} 2\pi i \,\frac{(x-i)}{0!}\,\frac{e^{ix}}{(x+i)(x-i)}\Big|_{x=i} = \frac{e^{i\cdot i}}{2i}\cdot 2\pi i = \frac{\pi}{e} \quad \text{for } R\to\infty$$

Parametrize with R or why $R\to\infty$?

Now we take a look at

$$\int_{\gamma_2} \frac{e^{ix}}{x^2+1}\,dx = \int_0^1 dt\, \frac{e^{iRe^{i\pi t}}}{R^2 e^{2\pi i t}+1}\cdot\left(Ri\pi e^{i\pi t}\right)$$

$x = Re^{i\pi t}$, $\frac{dx}{dt} = R(i\pi)e^{i\pi t}$

$$= \int_0^1 dt\,\left(R\pi i e^{i\pi t}\right)\frac{e^{iR(\cos(\pi t)+i\sin(\pi t))}}{R^2 e^{2\pi i t}+1}$$

$$= i\pi\int_0^1 dt\, \underbrace{\frac{R\,e^{R(i\cos(\pi t)-\sin(\pi t))+i\pi t}}{R^2 e^{2\pi i t}+1}}$$

$$\xrightarrow{R\to\infty} \frac{1}{Re^{2\pi i t}}\, e^{-R\sin(\pi t)}\cdot e^{i\pi t} = \frac{e^{-R\sin(\pi t)}}{Re^{i\pi t}} \xrightarrow{R\to\infty} 0$$ ✓ ⊕

What about the oscillating term $Ri\cos(\pi t)$?

c) $\langle 0|[\phi(x),\phi(y)]|0\rangle = D(x-y) - D(y-x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$

$\overset{!}{=} D_R(x-y) \quad \text{for } x_0 > y_0$

Wieso will man vom 3dim. Integral auf 4dim.?

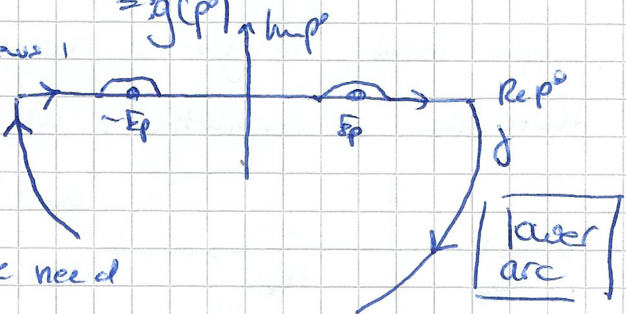
$$D_R(x-y) = \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{-1}{(p^0)^2-\mathbf{p}^2-m^2} e^{-i(p^0(x^0-y^0)-\mathbf{p}(\mathbf{x}-\mathbf{y}))}$$

$(p^0)^2-\mathbf{p}^2-m^2 = (p^0)^2-(\mathbf{p}^2+m^2) \rightarrow$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi i} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \int_{-\infty}^{\infty} dp^0 \underbrace{\frac{-1}{(p^0-\sqrt{\mathbf{p}^2+m^2})(p^0+\sqrt{\mathbf{p}^2+m^2})} e^{-ip^0\overbrace{(x^0-y^0)}^{>0}}}_{=g(p^0)}$$

We will now integrate as follows:



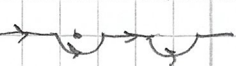
because we have

$e^{-ip^0(x^0-y^0)} = e^{-i(x^0-y^0)(\mathrm{Re}\,p^0 + i\,\mathrm{Im}\,p^0)}$

$= e^{-(x^0-y^0)(i\,\mathrm{Re}\,p^0 - \mathrm{Im}\,p^0)}$ and we need

an exponential decay in the integrand to have a vanishing contribution (for $\mathrm{Im}\,p^0 < 0$, this is given) from the complex plane (otherwise there would be a singularity at $p^0 \rightarrow +i\infty$)

Was wenn ich



so integriere?

Shouldn't we shift the whole function, not just part of the contour?

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi i} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \int_\gamma dp^0\, g(p^0)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi i} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} (2\pi i) \sum_{x_0} \mathrm{Res}(g, x_0)\, I(\gamma, x_0)$$

both times negative, as we encircle the poles clockwise

$$\mathrm{Res}(g, +\sqrt{\mathbf{p}^2+m^2}) = \frac{-e^{-ip^0(x^0-y^0)}}{p^0+\sqrt{\mathbf{p}^2+m^2}}\bigg|_{p^0=\sqrt{\mathbf{p}^2+m^2}}$$

$$= \frac{-e^{-iE_p(x^0-y^0)}}{2E_p} \quad \text{with } E_p^2 = \mathbf{p}^2+m^2$$

$$\mathrm{Res}(g, -\sqrt{\mathbf{p}^2+m^2}) = \frac{-e^{-ip^0(x^0-y^0)}}{p^0-\sqrt{\mathbf{p}^2+m^2}}\bigg|_{p^0=-\sqrt{\mathbf{p}^2+m^2}}$$

$$= \frac{e^{iE_p(x^0-y^0)}}{2E_p}$$

Difference to Feynman Propagator? Shouldn't we always get the same result from (10), independent of integration?

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \left(e^{-iE_p(x^0-y^0)} - e^{iE_p(x^0-y^0)}\right) \cdot \frac{1}{2E_p}$$

$\mathbf{p} \mapsto -\mathbf{p}$ for 2^{nd} integrand/integral $\rightsquigarrow$ no change in sign $\rightarrow$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip(x-y)} - e^{-ip(y-x)}\right) = D(x-y) - D(y-x)$$

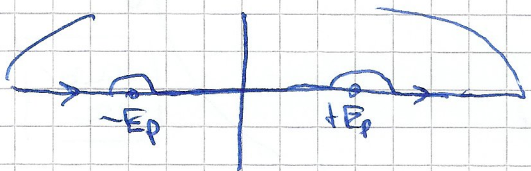
Now we take a look at $D_R(x-y)$ for $x_0 < y_0$

$$D_R(x-y) = \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip(x-y)}$$

$$= \dots = \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{+ip^0(y^0-x^0)} e^{-i\vec{p}(\vec{x}-\vec{y})}, \text{ because } y^0 > x^0$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\pi i} e^{-i\vec{p}(\vec{x}-\vec{y})} \int_{-\infty}^{\infty} dp^0 \frac{-1}{p^2-m^2} e^{ip^0(y^0-x^0)}$$

For the same contour, we now have to close the arc in the upper half to get a vanishing exponential for $\mathrm{Im}\, p^0 > 0$ and therefore no contribution. We thus integrate like this:



and the Residue theorem gives rise to a sum over no residues in this area

$\rightsquigarrow D_R(x-y) = 0$ for $x_0 < y_0$

This preserves causality, as a particle can only get from y to x, if $y_0 < x_0$ and would have to go back in time for $x_0 < y_0$

Why retarded

To see it's the greens function of the KG operator, we look at:

$$(\partial_x^2 + m^2) D_F(x-y) = \int \frac{d^4p}{(2\pi)^4 i} \frac{-1}{p^2-m^2} (-p^2+m^2) e^{-ip(x-y)}$$

4

$$= \int \frac{d^4p}{(2\pi)^4 i} e^{-ip(x-y)} = -i\,\delta(x-y)$$

representation of the δ-Distribution