Disclaimer

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19.05.2017 Quantum Field theory Exercise 4 Marvin Zanke

#4)

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

a) E.L. eq: $\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} - \frac{\delta \mathcal{L}}{\delta \psi} = 0$

More than 1 component? $\leadsto$ 4 eq. $\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi_a)} \ldots$

$$\Leftrightarrow 0 = \partial_\mu(\bar{\psi} i\gamma^\mu) + \bar{\psi} m = \bar{\psi}(\overleftarrow{\partial_\mu} i\gamma^\mu + m)$$

$$= \bar{\psi}(i\overleftarrow{\not{\partial}} + m)$$

analogue for $\bar{\psi}$:

$$0 = \partial_\mu(0) - (i\gamma^\mu\overrightarrow{\partial_\mu} - m)\psi$$

$$\Leftrightarrow (-i\gamma^\mu\overrightarrow{\partial_\mu} + m)\psi = 0$$

$$\Leftrightarrow (i\not{\partial} - m)\psi = 0$$

b)
$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi - g^{\mu\nu}\mathcal{L}$$

$$= \bar{\psi} i\gamma^\mu(\partial^\nu\psi) - g^{\mu\nu}\bar{\psi}\underbrace{(i\not{\partial} - m)\psi}$$

use eq. of motion (Dirac) here $\leadsto$ vanishes

$$= i\bar{\psi}\gamma^\mu(\partial^\nu\psi)$$

general energy momentum tensor (derived with infinitesimal translation) needs $\mathcal{L}$ to be independent of space-time?

c) Let ψ be a solution of the Dirac eq. for massless fermions, such that $(i\not{\partial} - m)\psi = i\not{\partial}\psi = 0$

$$\leadsto i\not{\partial}(\gamma^5\psi) = i\partial_\mu\gamma^\mu\gamma^5\psi = -i\partial_\mu\gamma^5\gamma^\mu\psi$$

$$= -i\gamma^5\not{\partial}\psi = 0$$

where we used $\{\gamma^5, \gamma^\mu\} = \{i\gamma^0\gamma^1\gamma^2\gamma^3, \gamma^\mu\}$

$$= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3) = 0$$

All but one permutation yields a $(-)$
$\leadsto 3\times(-1)$

d) $P_{L/R} := \frac{1}{2}(\mathbb{1} \mp \gamma^5)$

i) $P_{L/R}^2 = \frac{1}{4}(\mathbb{1} \mp \gamma^5)^2 = \frac{1}{4}(\mathbb{1} + (\gamma^5)^2 \mp 2\gamma^5) = \frac{1}{4}(2\mathbb{1} \mp 2\gamma^5) = \underset{P_{L/R}}{\frac{1}{2}(\mathbb{1} \mp \gamma^5)}$

where we have used $(\gamma^5)^2 = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^0\gamma^1\gamma^2\gamma^3) = (\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3$

$$= (\gamma^1)^2\gamma^2\gamma^3\gamma^2\gamma^3 = (\gamma^2)^2(\gamma^3)^2 = \mathbb{1}$$

$$P_L P_R = \frac{1}{4}(\mathbb{1}-\gamma^5)(\mathbb{1}+\gamma^5) = \frac{1}{4}(\mathbb{1}-(\gamma^5)^2) = 0$$
$$P_R P_L = \frac{1}{4}(\mathbb{1}+\gamma^5)(\mathbb{1}-\gamma^5) = \frac{1}{4}(\mathbb{1}-(\gamma^5)^2) = 0$$

Noch was zeigen?

$$P_L + P_R = P_R + P_L = \frac{1}{2}(\mathbb{1}-\gamma^5) + \frac{1}{2}(\mathbb{1}+\gamma^5) = \mathbb{1}$$

iii)
$$\psi_{L/R} := P_{L/R}\psi \rightsquigarrow \psi = P_L\psi + P_R\psi$$

$$\rightsquigarrow \mathcal{L} = \bar{\psi}(i\not{\partial}-m)\psi = (\bar{\psi}_L + \bar{\psi}_R)(i\not{\partial}-m)(\psi_L+\psi_R)$$

(P_L+P_R) (P_L+P_R) $\bar{\psi}P_{L/R} \hat{=} \overline{(P_{L/R}\psi)} = \psi_{L/R}^{\dagger}\gamma^0 = \bar{\psi}_{L/R}$

$P_{L/R}$ acting on $\bar{\psi}$ defined like this?
$\rightsquigarrow \bar{\psi} = \psi^{\dagger}\gamma^0$
$\bar{\psi}P_{L/R} = \psi^{\dagger}\gamma^0 P_{L/R}$
$= \psi^{\dagger}P_{R/L}\gamma^0$
$= (P_{R/L}\psi)^{\dagger}\gamma^0$
$= \psi_{R/L}^{\dagger}\gamma^0$
$= \bar{\psi}_{R/L}$

$$= (\bar{\psi}_L + \bar{\psi}_R)i\not{\partial}(\psi_L+\psi_R) - m(\bar{\psi}_L+\bar{\psi}_R)(\psi_L+\psi_R)$$

$$= \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_L i\not{\partial}\psi_R + \bar{\psi}_R i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R$$
$$- m(\bar{\psi}_L\psi_L + \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L + \bar{\psi}_R\psi_R)$$

$$\bar{\psi}_L i\not{\partial}\psi_R = \left(\frac{1-\gamma^5}{2}\psi\right)^{\dagger}\gamma^0 i\partial_\mu\gamma^\mu\left(\frac{1+\gamma^5}{2}\psi\right)$$
$$\overset{\text{def. of } P_L \text{ and } P_R,\ \{\gamma^5,\gamma^\mu\}=0}{=} \psi^{\dagger}\gamma^0\frac{1+\gamma^5}{2}i\partial_\mu\gamma^\mu\frac{1+\gamma^5}{2}\psi = \bar{\psi}i\partial_\mu\gamma^\mu\underbrace{\left(\frac{1-\gamma^5}{2}\right)\left(\frac{1+\gamma^5}{2}\right)}_{P_LP_R=0}\psi = 0$$

$$\bar{\psi}_R i\not{\partial}\psi_L = \left(\frac{1+\gamma^5}{2}\psi\right)^{\dagger}\gamma^0 i\partial_\mu\gamma^\mu\left(\frac{1-\gamma^5}{2}\psi\right)$$
$$\overset{\text{analogue}}{=} \bar{\psi}i\partial_\mu\gamma^\mu\underbrace{\left(\frac{1+\gamma^5}{2}\right)\left(\frac{1-\gamma^5}{2}\right)}_{P_RP_L}\psi = 0$$

$$= \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R$$
$$+ m(\underbrace{\bar{\psi}_L\psi_L}_{0} + \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L + \underbrace{\bar{\psi}_R\psi_R}_{0})$$

$\bar{\psi}_L\psi_R \neq 0$ but what about $\bar{\psi}_L\psi_L$? Analogous $\rightsquigarrow \bar{\psi}_L\psi_L = \bar{\psi}\underbrace{P_RP_L}_{=0}\psi = 0$

$$= \underbrace{\bar{\psi}_L i\not{\partial}\psi_L}_{\mathcal{L}_L} + \underbrace{\bar{\psi}_R i\not{\partial}\psi_R}_{\mathcal{L}_R} \quad \text{for } m=0$$

So Lagrange density decomposes into left- and right handed part for massless fermions. Otherwise we would still have the coupling with the mass between them.

Explicitly prove that not possible for $m\neq 0$?

e)
$$\begin{pmatrix}\psi \\ \psi^\dagger\end{pmatrix} \longmapsto \begin{pmatrix}\gamma^5\psi \\ \psi^\dagger\gamma^5\end{pmatrix} \rightsquigarrow \begin{pmatrix}\psi \\ \bar\psi\end{pmatrix} = \begin{pmatrix}\psi \\ \psi^\dagger\gamma^0\end{pmatrix} \longmapsto \begin{pmatrix}\gamma^5\psi \\ \psi^\dagger\gamma^5\gamma^0\end{pmatrix}$$

$$\bar\psi\psi \longmapsto \psi^\dagger\gamma^5\gamma^0\gamma^5\psi \overset{\{\gamma^5,\gamma^\mu\}=0}{=} -\psi^\dagger(\gamma^5)^2\gamma^0\psi \overset{(\gamma^5)^2=\mathbb{1}}{=} -\bar\psi\psi$$

what for? Physical meaning?

$\rightsquigarrow$ not invariant

$$\bar\psi i\gamma^5\psi \longmapsto \psi^\dagger\gamma^5\gamma^0 i\gamma^5\gamma^5\psi = \psi^\dagger\gamma^5\gamma^0 i\psi = -\psi^\dagger\gamma^0 i\gamma^5\psi = -\bar\psi i\gamma^5\psi$$

$\rightsquigarrow$ not invariant

$$\bar\psi\gamma^\mu\psi \longmapsto \psi^\dagger\gamma^5\gamma^0\gamma^\mu\gamma^5\psi = -\psi^\dagger\gamma^0\gamma^5\gamma^\mu\gamma^5\psi = \bar\psi\gamma^\mu\psi$$

$\rightsquigarrow$ invariant

$$\bar\psi\gamma^\mu\gamma^5\psi \longmapsto \psi^\dagger\gamma^5\gamma^0\gamma^\mu\gamma^5\gamma^5\psi = -\psi^\dagger\gamma^0\gamma^5\gamma^\mu\psi = \bar\psi\gamma^\mu\gamma^5\psi$$

$\rightsquigarrow$ invariant

$$\bar\psi\sigma^{\mu\nu}\psi \longmapsto \psi^\dagger\gamma^5\gamma^0\sigma^{\mu\nu}\gamma^5\psi \overset{\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]}{=} -\psi^\dagger\gamma^0\gamma^5\frac{i}{2}[\gamma^\mu,\gamma^\nu]\gamma^5\psi$$
$$= -\bar\psi\frac{i}{2}[\gamma^\mu,\gamma^\nu](\gamma^5)^2\psi = -\bar\psi\sigma^{\mu\nu}\psi$$

$\rightsquigarrow$ not invariant

f)
$$V^\mu(x) := \bar\psi(x)\gamma^\mu\psi(x)$$
$$\partial_\mu V^\mu(x) = \partial_\mu(\bar\psi(x)\gamma^\mu\psi(x)) = \bar\psi(x)\overleftarrow{\partial\!\!\!/}\psi(x) + \bar\psi(x)\overrightarrow{\partial\!\!\!/}\psi(x)$$

Do $\psi, \bar\psi$ fulfill Dirac eq. now?

$\rightsquigarrow$ Yes! Problem is that before, we were looking for solutions of ψ by demanding a minimum of Action and thus couldn't already use Dirac eq.

$\bar\psi(i\partial\!\!\!/+m)=0 \Leftrightarrow \bar\psi\partial\!\!\!/ = -\frac{m}{i}\bar\psi$ | $(i\partial\!\!\!/-m)\psi=0 \Leftrightarrow \partial\!\!\!/\psi = \frac{m}{i}\psi$

$$= \left(-\frac{m}{i}\bar\psi(x)\right)\psi(x) + \bar\psi(x)\left(\frac{m}{i}\psi(x)\right) = 0$$

$$A^\mu(x) := \bar\psi(x)\gamma^\mu\gamma^5\psi(x)$$
$$\partial_\mu A^\mu(x) = \partial_\mu(\bar\psi(x)\gamma^\mu\gamma^5\psi(x)) = \bar\psi(x)\overleftarrow{\partial\!\!\!/}\gamma^5\psi(x) + \bar\psi(x)\overrightarrow{\partial\!\!\!/}\gamma^5\psi(x)$$
$$= \bar\psi(x)\overleftarrow{\partial\!\!\!/}\gamma^5\psi - \bar\psi(x)\gamma^5\overrightarrow{\partial\!\!\!/}\psi(x)$$
$$\overset{\text{Dirac}}{=} \left(-\frac{m}{i}\bar\psi(x)\right)\gamma^5\psi(x) - \bar\psi(x)\gamma^5\left(\frac{m}{i}\psi(x)\right)$$
$$= -2\frac{m}{i}\bar\psi(x)\gamma^5\psi(x) \neq 0 \text{ for } m\neq 0$$
$$= 0 \text{ for } m=0$$

Could also account to zero?

Now, we look at the transformation $\psi(x) \mapsto e^{i\alpha}\psi(x) \rightsquigarrow \bar{\psi}(x) \mapsto e^{-i\alpha}\bar{\psi}(x)$

It's associated Noether current is $j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \psi)} \Delta\psi - X^\mu$

Where $X^\mu = 0$, as $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \mapsto e^{-i\alpha}\bar{\psi}(i\not{\partial}-m)e^{i\alpha}\psi$

$= \mathcal{L}$

The infinitesimal transformation is $\psi'(x) \approx (1+i\alpha)\psi(x)$

$\bar{\psi}'(x) \approx (1-i\alpha)\bar{\psi}(x)$

$\rightsquigarrow \alpha\Delta\psi'(x) = i\alpha\psi(x) \Leftrightarrow \Delta\psi(x) = i\psi(x)$

$\alpha\Delta\bar{\psi}(x) = -i\alpha\bar{\psi}(x) \Leftrightarrow \Delta\bar{\psi}(x) = -i\bar{\psi}(x)$

$\rightsquigarrow j^\mu = \bar{\psi}(x) i\gamma^\mu i\psi(x) = -\bar{\psi}(x)\gamma^\mu\psi(x) = -V^\mu(x)$ ✓

g) $\mathcal{L}(x) \mapsto \mathcal{L}(x) + \alpha\partial_\mu X^\mu(x)$ doesn't affect the action and therefore leaves the field equations invariant

„An" appropriate 4-divergence $\rightsquigarrow$ more possible

Postulate $X^\mu(x) = \frac{1}{2}(\bar{\psi}(x) i\gamma^\mu\psi(x))$

$\rightsquigarrow$ we even know that $\partial_\mu X^\mu(x) \stackrel{f)}{=} 0$, so it's already a symmetry of the Lagrangian

$\rightsquigarrow \tilde{\mathcal{L}}(x) = \bar{\psi}(x)(i\not{\partial}-m)\psi(x) - \alpha\partial_\mu \frac{1}{2}(\bar{\psi}(x) i\gamma^\mu\psi(x))$

$= \bar{\psi}(x)\left(\frac{i\overrightarrow{\not{\partial}}}{2} - m\right)\psi(x) - \bar{\psi}(x)\frac{i\overleftarrow{\not{\partial}}}{2}\psi(x)$

What happens to the α? $\rightsquigarrow$ choose $\alpha = 1$

$\rightsquigarrow \tilde{T}^{\mu\nu} = \frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\psi)}\partial^\nu\psi + (\partial^\nu\bar{\psi})\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu\bar{\psi})} - g^{\mu\nu}\tilde{\mathcal{L}}$

$(\partial^\nu\bar{\psi})\frac{\delta\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}$ instead of $\frac{\delta\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}(\partial^\nu\bar{\psi})$? $\rightsquigarrow$ order important

$= \bar{\psi}(x)\frac{i\gamma^\mu}{2}(\partial^\nu\psi) + (\partial^\nu\bar{\psi})\left(-\frac{i\gamma^\mu}{2}\psi(x)\right)$

$- g^{\mu\nu}\Big\{\underbrace{\bar{\psi}(x)(i\not{\partial}-m)\psi(x)}_{=0,\text{ as }\psi\text{ solves Dirac eq.}} - \underbrace{\frac{1}{2}\partial_\kappa(\bar{\psi}(x) i\gamma^\kappa\psi(x))}_{=0,\text{ see f)}}\Big\}$

$= \frac{i}{2}\bar{\psi}(x)\gamma^\mu(\overrightarrow{\partial^\nu} - \overleftarrow{\partial^\nu})\psi(x)$ ✓

E.L. eq. $0 = \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)} - \frac{\delta\mathcal{L}}{\delta\psi} = \partial_\mu\bar{\psi}(x)\frac{i}{2}\gamma^\mu + \bar{\psi}(x)m + \bar{\psi}(x)\frac{i\overleftarrow{\not{\partial}}}{2}$

$= \bar{\psi}(x)(i\overleftarrow{\not{\partial}} + m)$ ✓

$0 = \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})} - \frac{\delta\mathcal{L}}{\delta\bar{\psi}} = \partial_\mu\left(-\frac{i}{2}\gamma^\mu\psi(x)\right) - \left(\frac{i\overrightarrow{\not{\partial}}}{2} - m\right)\psi(x)$

$= -i\overrightarrow{\not{\partial}}\psi(x) + m\psi(x)$

$\Leftrightarrow 0 = (i\not{\partial} - m)\psi(x)$ ✓