29.05.2017 Quantum Field Theory Exercise 5 Marvin Zanke

H5) $$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left[a_s(p) u_s(p) e^{-ipx} + b_s^\dagger(p) v_s(p) e^{ipx} \right]$$

hy does P
have no effect
on $u_s(p)$ etc.?
⤳ acting on Hilbert space
(operators acting on H.S.)

$$P\Psi(\underset{\sim}{x},t)P^{-1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ P a_s(p) P^{-1} u_s(p) e^{-ipx} + P b_s^\dagger(p) P^{-1} v_s(p) e^{ipx}$$

$\Psi(x)$ doesn't
consist of
$a_s^\dagger$?

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ \eta_a a_s(-p) \underset{\uparrow}{u_s(p)} e^{-ipx} + \eta_b^* b_s^\dagger(-p) \underset{\uparrow}{v_s(p)} e^{ipx}$$

$\gamma^0\gamma^0 = 1$ $\gamma^0\gamma^0 = 1$

$$P b_s(p) P^{-1} = \eta_b b_s(-p)$$

$$\overset{()^\dagger}{\Longleftrightarrow} P b_s^\dagger(p) P^{-1} = \eta_b^* b_s^\dagger(-p)$$

$P^\dagger = P = P^{-1}$

and $P a_s(p) P^{-1} = \eta_a a_s(-p)$

$P = P^{-1}$ for
$P = e^{i\varphi}\gamma^0$?
⤳ $P^\dagger = P^{-1}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ \eta_a a_s(p) \gamma^0 u_s(-p) e^{-ipx} + \eta_b^* b_s^\dagger(-p) \gamma^0 (-v_s(-p)) e^{ipx} \}$$

$\uparrow$
$\gamma^0 u_s(p) = u_s(-p)$ and $\gamma^0 v_s(p) = -v_s(-p)$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ \eta_a \gamma^0 a_s(-p) u_s(-p) e^{-ipx} - \eta_b^* \gamma^0 b_s^\dagger(-p) v_s(-p) e^{ipx} \}$$

$\Psi(-\underset{\sim}{x},t) \rightarrow$
thus not needed?
⤳ 2 ways, either like
this or $p \to -p$ etc.

$$\overset{!}{=} M \Psi(-\underset{\sim}{x},t)$$

$= (\ldots)$ Zwischenschritt mit
$\vec{p} \to -\vec{p}$

determine
more than that?
⤳ 4th Ex. sheet
yielded $\gamma^0 u_s(\vec{p}) = u_s(-\vec{p})$
$\gamma^0 v_s(\vec{p}) = -v_s(-\vec{p})$
and thus $+1$ is
the right phase

± 1 but then use Ex. sheet 4.

$$\Longleftrightarrow -\eta_b^* = \eta_a \equiv \eta = 1, \quad \underline{M = \eta\gamma^0}$$

$$\leadsto P\Psi(\underset{\sim}{x},t)P^{-1} = \underline{\underline{\eta\gamma^0 \Psi(-\underset{\sim}{x},t)}}, \text{ as } \underset{\sim}{x} \longmapsto -\underset{\sim}{x} = \underset{\sim}{x}'$$

$$\Rightarrow p \longmapsto -p = p'$$

and $px = p^0x^0 - \underset{\sim}{p}\cdot\underset{\sim}{x}$
$= p^0x^0 - p'\cdot\underset{\sim}{x}'$

T on an
operator?
⤳ any on
$a^{(\dagger)}$ etc.

using $Tc = c^*T$

b) $$T\Psi(\underset{\sim}{x},t)T^{-1} \overset{\downarrow}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ T a_s(p) T^{-1} u_s^*(p) e^{+ipx} + T b_s^\dagger(p) T^{-1} v_s^*(p) e^{-ipx}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ (-)^{1/2+s} a_{-s}(-p) (i\gamma^0\gamma^2\gamma^5)(i\gamma^5\gamma^2\gamma^0) u_s^*(p) e^{ipx}$$
$$+ (-)^{1/2+s} b_{-s}^\dagger(-p) (i\gamma^0\gamma^2\gamma^5)(i\gamma^5\gamma^2\gamma^0) v_s^*(p) e^{-ipx}$$

why $P\Psi P$ troubles
on operators?
⤳ $\langle x|P\hat{x}P^{-1}|\Psi\rangle$
$= \langle -x|\hat{x}P^{-1}|\Psi\rangle$
$= -x\langle -x|P^{-1}|\Psi\rangle$
$= -x\langle x|\Psi\rangle$
$= -\langle x|x|\Psi\rangle$
[see paper in Theo 3 folder]

$$T b_s(p) T^{-1} = (-)^{1/2+s} b_{-s}(-p)$$
$$\Longleftrightarrow T b_s^\dagger(p) T^{-1} = (-)^{1/2+s} b_{-s}^\dagger(-p)$$

and $T a_s(p) T^{-1} = (-)^{1/2+s} a_{-s}(-p)$

and $(i\gamma^5\gamma^2\gamma^0)^{-1} = i\gamma^0\gamma^2\gamma^5$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s i\gamma^0\gamma^2\gamma^5 \{ (-)^{1/2+s} a_{-s}(-p) (-)^{1/2-s} u_{-s}(-p) e^{ipx} + (-)^{1/2+s} b_{-s}^\dagger(-p) (-)^{1/2-s} v_{-s}(-p) e^{-ipx}$$

Using $i\gamma^5\gamma^2\gamma^0 u_s(p) = (-)^{1/2-s} u^*_{-s}(-p)$

$(\gamma^2)^* = -\gamma^2$ $\Longleftrightarrow -i(\gamma^5)^*(\gamma^2)^*(\gamma^0)^* u^*_s(p) = (-)^{1/2-s} u_{-s}(-p)$ and $i\gamma^5\gamma^2\gamma^0 v^*_s(p) = (-)^{1/2-s} v_s(-p)$

all others $(\gamma^\sigma)^* = \gamma^\sigma$ $\Longleftrightarrow i\gamma^5\gamma^2\gamma^0 u^*_s(p) = (-)^{1/2-s} u_s(-p)$ analogue

Why can we exactly determine phase here

$$= -i\gamma^0\gamma^2\gamma^5 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ a_{-s}(-p) u_{-s}(-p) e^{ipx} + b^\dagger_{-s}(-p) v_{-s}(-p) e^{-ipx} \}$$

$$\overset{!}{=} M \psi(\underline{x}, -t)$$

$$\Longleftrightarrow M = -i\gamma^0\gamma^2\gamma^5 = -i\gamma^0\gamma^1\gamma^1\gamma^3\gamma^3\gamma^2\gamma^5 = -i\gamma^0\gamma^1\gamma^3\gamma^1\gamma^2\gamma^3\gamma^5$$

$$= -i\gamma^1\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = -\gamma^1\gamma^3$$

Peskin: $M = \gamma^1\gamma^3$?
⇝ different repr. ?

as $t \longmapsto -t = t' \Rightarrow p \longmapsto -p = p'$

$s \longmapsto -s = s'$

and $p\cdot x = p^0 t - p\cdot x = -(p^0 t' - p'\cdot x') = -p'\cdot x'$ ✓

$$\leadsto T\psi(\underline{x}, t)T^{-1} = -\gamma^1\gamma^3\psi(\underline{x}, -t)$$

$C = C^{-1} = C^\dagger$?

c) $$C\psi(\underline{x}, t)C^{-1} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ C a_s(p) C^{-1} u_s(p) e^{-ipx} + C b^\dagger_s(p) C^{-1} v_s(p) e^{ipx} \}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ b_s(p)\, i\gamma^2 i\gamma^2 u_s(p) e^{-ipx} + a^\dagger_s(p)\, i\gamma^2 i\gamma^2 v_s(p) e^{ipx} \}$$

$C b_s(p) C^{-1} = a_s(p)$ and $C a_s(p) C^{-1} = b_s(p)$

$\Longleftrightarrow C b^\dagger_s(p) C^{-1} = a^\dagger_s(p)$ and $(i\gamma^2)^{-1} = i\gamma^2$

$$= i\gamma^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \{ b_s(p)(-v^*_s(p)) e^{-ipx} + a^\dagger_s(p)(-u^*_s(p)) e^{ipx} \}$$

$$= -i\gamma^2 \psi^*(\underline{x}, t) = M\psi^*(\underline{x}, t)$$

in AQT $i\gamma^2$ without the $(-)$?

$$\leadsto M = -i\gamma^2$$ ✓

✓

$b^\dagger_s = b^*_s$?
⇝ in general Matrix in Basis

T is antiunitary: $\langle \phi | T^\dagger \psi \rangle = \langle T\phi | \psi \rangle^*$ and
antilinear: $T(\eta_1\phi + \eta_2\psi) = \eta_1^* T\phi + \eta_2^* T\psi$

T antiunitary because
$$T\psi(x,t)T^{-1} = \underbrace{T e^{-iHt} T^{-1}}_{e^{iHt}} \underbrace{T \psi(x,0) T^{-1}}_{M\psi(x,0)}$$
$$= M\psi(x, -t)$$

if T was unitary ⇝ $Te^{-iHt}T^{-1} = e^{iHt}$

$\Rightarrow \{T, H\} = 0$, $H(T|\psi(x,t)\rangle) = -TH|\psi(x,t)\rangle$

$= -ET|\psi(x,t)\rangle$ ⇝ H unbound from below

⇝ T is a

$\Rightarrow [T, H] = 0$

H6)

$$|\Psi, S=0,1\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\left\{ a^\dagger_{1/2}(p) b^\dagger_{-1/2}(-p) + (-)^{S-1} a^\dagger_{-1/2}(p) b^\dagger_{1/2}(-p)\right\} \times |0\rangle$$

Only apply P from left to state? ✓

↳ it's an ELEMENT of Hilbert space and no operator

$$PP = \mathbb{1} \text{ and } P|0\rangle = |0\rangle$$

$$P|\Psi, S=0,1\rangle \overset{PP=\mathbb{1}}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\left\{ P a^\dagger_{1/2}(p) PP b^\dagger_{-1/2}(-p) P + (-)^{S-1} P a^\dagger_{-1/2}(p) PP b^\dagger_{1/2}(-p) P \right\} |0\rangle$$

m=0 and no coupling to J=L+S

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\left\{ \eta_a^* \eta_b^* a^\dagger_{1/2}(-p) b^\dagger_{-1/2}(p) + (-1)^{S-1} \eta_a^* \eta_b^* a^\dagger_{-1/2}(-p) b^\dagger_{1/2}(p)\right\} |0\rangle$$

Why anti-symmetric for S=0? ✓

$(-)^{S-1}$?

⇝ Clebsch-Gordan

z.B. $|1,1\rangle = |1/2, 1/2\rangle \times |1/2, 1/2\rangle$, $|1,0\rangle = 1/\sqrt{2}\,(|1/2, 1/2\rangle \otimes |1/2, -1/2\rangle + |1/2, -1/2\rangle \otimes |1/2, 1/2\rangle$

$p \mapsto -p$

$$\overset{\downarrow}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\, \eta_a^* \eta_b^* \left\{ a^\dagger_{1/2}(p) b^\dagger_{-1/2}(-p) + (-)^{S-1} a^\dagger_{-1/2}(p) b^\dagger_{1/2}(-p)\right\} |0\rangle$$

Opposite p and S?

$$= \eta_a^* \eta_b^* |\Psi, S=0,1\rangle \underset{\text{from H5}}{\overset{\eta_b^* = -\eta_a}{=}} -|\eta_a|^2 |\Psi, S=0,1\rangle$$

$P a_s(p) P^{-1} = \eta_a a_s(-p)$ independent of s?

$$= -|\Psi, S=0,1\rangle \quad \text{with} \quad |\eta_{a,b}|^2 = 1$$

✓

Using H5 here $\eta_b^* = -\eta_a$?

b)
$$C|\Psi, S=0,1\rangle \underset{\substack{\text{and } C|0\rangle = |0\rangle \\ C^{-1}|0\rangle = |0\rangle}}{\overset{C^2=1}{=}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\left\{ C a^\dagger_{1/2}(p) C^{-1} C b^\dagger_{-1/2}(-p) C^{-1} + (-)^{S-1} C a^\dagger_{-1/2}(p) C^{-1} C b^\dagger_{1/2}(-p) C^{-1}\right\} |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\left\{ b^\dagger_{1/2}(p)\, a^\dagger_{-1/2}(-p) + (-)^{S-1} b^\dagger_{-1/2}(p)\, a^\dagger_{1/2}(-p)\right\} |0\rangle$$

$$= (-)^{S-1} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\left\{ (-)^{S-1} b^\dagger_{1/2}(p) a^\dagger_{-1/2}(-p) + b^\dagger_{-1/2}(p) a^\dagger_{1/2}(-p)\right\} |0\rangle$$

$$\underset{\text{anticom. rel.}}{\overset{\{a^\dagger, b^\dagger\}=0}{=}} (-)^{S-1} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|) (-1) \left\{ a^\dagger_{1/2}(-p) b^\dagger_{-1/2}(p) + (-)^{S-1} a_{-1/2}(-p) b^\dagger_{1/2}(p)\right\} |0\rangle$$

$$\overset{p \mapsto -p}{=} (-)^S \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Phi_S(|p|)\left\{ a^\dagger_{1/2}(p)\, b^\dagger_{-1/2}(-p) + (-)^{S-1} a^\dagger_{-1/2}(p)\, b^\dagger_{1/2}(-p)\right\} |0\rangle$$

$$= (-)^S |\Psi, S=0,1\rangle$$

✓

c) We have $C A_\mu C^{-1} = -A_\mu$ and thus construct

$|n\rangle = A_{\mu_1} A_{\mu_2} \cdots |0\rangle$ ← $\epsilon^{\mu_1}\epsilon^{\mu_2}\dots$

$\rightsquigarrow C|n\rangle = C A_{\mu_1} C^{-1} C A_{\mu_2} C^{-1} \cdots |0\rangle$
$= (-1)^n A_{\mu_1} A_{\mu_2} \cdots |0\rangle = (-1)^n |n\rangle$ ✓

$\rightsquigarrow C|n\rangle = C|\psi, S=0,1\rangle \; (= (-)^S |\psi, S=0,1\rangle)$

$\overset{!}{\Leftrightarrow} |n\rangle = |\psi, S=0,1\rangle$ ✓

$\Leftrightarrow (-1)^{S+n} = 1 \Leftrightarrow n+S = 2k \Leftrightarrow n$ and S odd or even

it is not $C|n\rangle = C|\psi, S=0,1\rangle$ only the Eigenvalues have to be equal $(-1)^n = (-1)^S$

Aufschreiben? $\rightsquigarrow |\psi, S=0,1\rangle \to |n\rangle$

$\rightsquigarrow S=0: \; n = 2, 4, \dots$
$S=1: \; n = 1, 3, \dots$
↑ not allowed, at least two photons (mom. conservation)

$\rightsquigarrow$ $S=0$ decay is more likely, as decays into fewer photons are more likely. ✓

$\rightsquigarrow$ consistent with $\tau_{S=0} = 1{,}25 \cdot 10^{-10}\,s << 1{,}41 \cdot 10^{-7}\,s = \tau_{S=1}$

more exact? $\rightsquigarrow S=0$ much smaller lifetime

Why charge conj. negative? $\rightsquigarrow E, B$ field change under $C \rightsquigarrow F_{\mu\nu}$ and A need to change

Momentum conservation because photon? ✓
$p_\gamma = \begin{pmatrix} 2m_e \\ 0 \end{pmatrix} \overset{!}{=} p_\gamma$
$\Leftrightarrow 4m_e^2 = p_\gamma^2 = 0$ ↯