Disclaimer

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Why no complex k.G. field?

hängt immer davon ab welche Teilchen du betrachten willst. ~~[illegible]~~

H7) $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi(x) \partial^\mu \phi(x) - m^2 \phi^2(x)\right) + j(x)\phi(x)$

$$H_I = -\int d^3x \, j(\underline{x}, t)\phi_I(\underline{x}, t)$$

a)

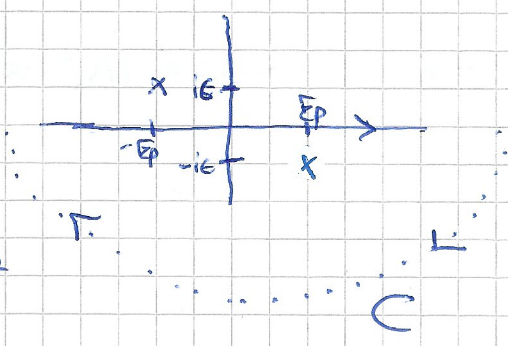
First, we take a look at

Fourier transform of the current $j(x)$

$$\lambda = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} |\tilde{j}(p)|^2$$

$$p^2 - m^2 + i\epsilon = (p^0)^2 - \underline{p}^2 - m^2 + i\epsilon = (p^0)^2 - E_{\underline{p}}^2 + i\epsilon = (p^0)^2 - (E_{\underline{p}}^2 - i\epsilon)$$

$$= \left(p^0 - (E_p - i\epsilon)\right)\left(p^0 + (E_p - i\epsilon)\right)$$



Why close in lower arc? Not clear what $|\tilde{j}(p)|^2$ is and therefore whether it vanishes stronger than $\frac{1}{(p^0)^2}$

Close integration contour in lower half, no contribution on arc for $|p^0| \to \infty$

$$= \int \frac{d^3p}{(2\pi)^3} \int_C \frac{dp^0}{2\pi} \frac{i}{(p^0 - (E_p - i\epsilon))(p^0 + (E_p - i\epsilon))} |\tilde{j}(p)|^2$$

use residue theorem, negative winding number, pole of order 1 at $(E_p - i\epsilon)$

$$= \int \frac{d^3p}{(2\pi)^4} i(-2\pi i) \left(\frac{1}{p^0 + (E_p - i\epsilon)}\right)\Big|_{p^0 = E_p - i\epsilon} |\tilde{j}(p)|^2\Big|_{p^0 = E_p - i\epsilon}$$ ✓

$$\overset{\epsilon \to 0}{=} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\tilde{j}(p)|^2 \quad \text{with } p^0 = E_p$$ ✓

Additionally

$$\tilde{j}(p) = \mathcal{F}_j(p) = \int j(k) e^{-ipk} d^4k = \int dk^0 d^3k \, j(k) e^{-ip^0k^0} e^{i\underline{p}\underline{k}}$$

derived in the Lecture

$$\Rightarrow |\tilde{j}(p)|^2 = \tilde{j}(p) \cdot \tilde{j}^*(p) = \int d^4k \, d^4l \, j(k) j^*(l) e^{-ipk} e^{ipl} = \int d^4k \, d^4l \, j(k) j(l) e^{ip(l-k)}$$

j is real, weil $\mathcal{L}$ wg. Unitarität reell sein muss

Now, finally:

Why the T in A?

$$A = \langle 0 | T \exp\left(-i \int dt \, H_I(t)\right) | 0 \rangle = \langle 0 | T \exp\left(+i \int d^4x \, j(x) \phi_I(x)\right) | 0 \rangle$$

$$= \langle 0 | T \left\{ 1 + i \int d^4x \, j(x) \phi_I(x) - \frac{1}{2} \int d^4x \, j(x) \phi_I(x) \int d^4y \, j(y) \phi_I(y) + \mathcal{O}(j^3) \right\} | 0 \rangle$$

$$\approx \langle 0 | 0 \rangle + i \langle 0 | \int d^4x \, T j(x) \phi_I(x) | 0 \rangle - \frac{1}{2} \langle 0 | \int d^4x \, d^4y \, T j(x) \phi_I(x) j(y) \phi_I(y) | 0 \rangle$$

$j(x)$ classical so T has no effect on it! (just a c-number)

$$= 1 + i \langle 0 | \int d^4x \, j(x) T \phi_I(x) | 0 \rangle - \frac{1}{2} \langle 0 | \int d^4x \, d^4y \, j(x) j(y) T \phi_I(x) \phi_I(y) | 0 \rangle$$ ✓

$$= 1 + i\int d^4x\, j(x) \langle 0| T\phi_I(x)|0\rangle - \frac{1}{2}\int d^4x\, d^4y\, j(x) j(y) \langle 0| T\phi_I(x)\phi_I(y)|0\rangle$$

wieso $j(x)$ aus $\langle 0..|$ rausziehen?
⇝ nur Zahlen, und keine Op ⇒ Kommut.

$$= 1 + i\int d^4x\, j(x) \langle 0| :\phi_I(x):|0\rangle - \frac{1}{2}\int d^4x\, d^4y\, j(x)j(y)\langle 0| :\phi_I(x)\phi_I(y): + \overline{\phi_I(x)\phi_I(y)}|0\rangle$$

| as $\langle 0| : \_\_ : |0\rangle$ always vanishes (normal ordered)

$$= 1 - \frac{1}{2}\int d^4x\, d^4y\, j(x)j(y)\, D_F(x-y) = 1 - \frac{1}{2}\int d^4x\, d^4y\, j(x) j(y) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

$$= 1 - \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} \int d^4x\, d^4y\, j(x) j(y)\, e^{-ip(x-y)} = 1 - \frac{1}{2}\lambda$$

↑ see previous page

$$\rightsquigarrow P(0\to 0) := |A|^2 \approx |1 - \tfrac{\lambda}{2}|^2 \approx 1 - \lambda$$

What for did we need $j(x)=0$ for $t \to \pm\infty$
⇝ need this for pert. theory?

b) $$A = \langle 0 | T \sum_{n=0}^{\infty} \frac{\left(i \int d^4x \, j(x) \phi_I(x)\right)^n}{n!} | 0 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \langle 0 | T j(x_1) \phi_I(x_1) \ldots j(x_n) \phi_I(x_n) | 0 \rangle$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, j(x_1) \ldots j(x_n) \langle 0 | T \phi_I(x_1) \ldots \phi_I(x_n) | 0 \rangle$$

odd n yield zero $$\sum_{k=0}^{\infty} \frac{i^{2k}}{(2k)!} \int d^4x_1 \ldots d^4x_{2k} \, j(x_1) \ldots j(x_{2k}) \underbrace{\langle 0 | T \phi_I(x_1) \ldots \phi_I(x_{2k}) | 0 \rangle}_{\text{Pairs of propagators}}$$

Why do all summands yield the same and therefore $\sum s \mapsto x \cdot s$?

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int d^4x_1 \ldots d^4x_{2k} \, j(x_1) \ldots j(x_{2k}) \sum_{\substack{\text{pairings} \\ \text{of fields } i}} D_F(x_{i_1} - x_{i_2}) \ldots D_F(x_{i_{2k-1}} - x_{i_{2k}})$$

$$\binom{2k}{2} \cdot \binom{2k-2}{2} \ldots \binom{2}{2}$$

Summing over this sum of products has

$$\frac{(2k)!}{(2k-2)! \, 2!} \frac{(2k-2)!}{(2k-4)! \, 2!} \ldots \frac{2!}{2! \, 0!} = \frac{(2k)!}{(2!)^k}$$ summands, which all yield the same, as it is symmetric in i. As the order we choose those pairs out of $\phi_I(x_1) \ldots \phi_I(x_{2k})$ is not important, we need to factor out $\frac{1}{k!} \rightsquigarrow \frac{(2k)!}{(2!)^k k!}$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int d^4x_1 \ldots d^4x_{2k} \, j(x_1) \ldots j(x_{2k}) \frac{(2k)!}{(2!)^k k!} D_F(x_1 - x_2) \ldots D_F(x_{2k-1} - x_{2k})$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2!)^k k!} \underbrace{\int d^4x_1 \, j(x_1) j(x_2) D_F(x_1 - x_2)}_{\lambda} \ldots \underbrace{\int d^4x_{2k} \, j(x_{2k-1}) j(x_{2k}) D_F(x_{2k-1} - x_{2k})}_{\lambda}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{2^k} \lambda^k = \sum_{k=0}^{\infty} \left(\frac{-\lambda}{2}\right)^k \frac{1}{k!} = e^{-\lambda/2}$$

$$\rightsquigarrow P(0 \to 0) = |A|^2 = \left(e^{-\lambda/2}\right)^2 = e^{-\lambda}$$