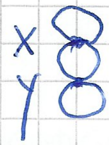
Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

21.06.2017 Quantum Field Theory Exercise sheet 7 Marvin Zanke

A7) i)



For ϕ^4-theory, the n-th term contributing to the transition amplitude is given by

$$A_n = \frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n \int d^4y_1 \dots d^4y_n \langle 0 | T\{\phi^4(y_1) \dots \phi^4(y_n)\} | 0\rangle$$

and therefore, we get for our case of 2 vertices,

$$A_2 = \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \int dy\, dx \langle 0 | T\{\phi(y)\phi(y)\phi(y)\underbrace{\phi(y)\phi(x)\phi(x)\phi(x)\phi(x)}_{\text{fixed}}\}|0\rangle$$

To get all permutations, we notice that in the diagram, we want x and y at least to be connected/contracted ~~once~~ twice, as well as x and y contracted with themselves once. This yields the contraction sketched in the time-ordered amplitude above as one example and also fixes the last contraction to be between x and y!

$\rightsquigarrow$ 2 x x contracted with y $\}$ 2 : there remain 2 possible ways to connect $\phi(x)\phi(y)$
1 x x contracted with x $\}\binom{4}{2}$: Chose 2 out of 4 $\phi(x)$ to contract them
1 x y contracted with y $\}\binom{4}{2}$: " — $\phi(y)$ — "

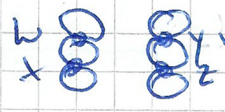
$$\rightsquigarrow \binom{4}{2}\binom{4}{2}\cdot 2 = \frac{(4!)^2\cdot 2}{(2!)^4}$$

"Start" from here

All in all, we get an inverse symmetry factor of

$$S_2^{-1} = \frac{1}{2!(4!)^2}\cdot 2\,\frac{(4!)^2}{(2!)^4} = \frac{1}{2^4} = \frac{1}{16}$$

$\rightsquigarrow$ Symmetry factor $S_2 = 16$

ii) Now the 2 disconnected diagrams  $\leadsto$ 4 vertices

$$\leadsto \frac{1}{4!}\left(\frac{-i\lambda}{4!}\right)^4 \int dw\,dx\,dy\,dz$$

$$\times \langle 0|T\{\phi(w)\phi(w)\phi(w)\phi(w)\phi(x)\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\}|0\rangle$$

- Choose 2 out of the 4 variables w,x,y,z, which you want to connect /contract in the left diagram, e.g. w,x

$\leadsto$ 3 possibilities $\left(\frac{\binom{4}{2}}{2!}\right)$

- between those 2 points and the 2 remaining points, it's the same possibilities as in i), i.e $\binom{4}{2}\binom{4}{2}\cdot 2$

$\leadsto \binom{4}{2}\cdot\binom{4}{2}\cdot 2$

$\Rightarrow$ all in all, we get

$$S_4^{-1} = \frac{4\cdot\binom{4}{2}\binom{4}{2}\binom{4}{2}\binom{4}{2}}{4!\,(4!)^4}\cdot 3 = \frac{\cancel{(4!)^4}\cdot 4}{(2!)^8\cdot 4!\,\cancel{(4!)^4}}\cdot 3 = \frac{1}{2!}\,\frac{1}{2^8}$$

$$= \frac{1}{2!}\,\frac{1}{16}\cdot\frac{1}{16} = \frac{1}{2!}\,\frac{1}{S_2^2}$$

$\leadsto S_4 = 512$

iii) We want to generalize the result of the example ii) to an arbitrary diagram of order N, which can be decomposed into m subdiagrams, each occuring n_i times and having the amplitude(s) X_i.

We want to start by proving the following:

$$\frac{1}{N!}\left(\frac{-i\lambda}{4!}\right)^N \int d^4y_1 \ldots d^4y_N \langle 0|T\{\phi^4(y_1)\ldots\phi^4(y_N)\}|0\rangle$$

$$\overset{!}{=} \frac{1}{n!}\underbrace{\left(\frac{1}{c!}\left(\frac{-i\lambda}{4!}\right)^c \int d^4y_1 \ldots d^4y_c \langle 0|T\{\phi^4(y_1)\ldots\phi^4(y_c)\}|0\rangle\right)}_{=:X}{}^n,$$

Say, that for one type of the m subdiagrams which occurs n times and each of those diagrams has c vertices, meaning a total number of vertices $N = n\cdot c$, the amplitude factorizes:

- Choose c vertices out of N, which „build" the first diagram $\rightsquigarrow \binom{N}{c}$

- Of the $(N-c)$ remaining, choose again c and repeat this until c left; those are immediately fixed

$$\rightsquigarrow \underbrace{\binom{N}{c}\binom{N-c}{c}\cdots\binom{2c}{c}}_{(n-1)\text{ times}}\cdot\cancel{\binom{c}{c}}$$

- divide by $\frac{1}{n!}$, as the order in which we choose those $(n-1)$ times (all in all n „pairs" fixed in the end) is not important $\longrightarrow$ Particularly, the permutations among $\binom{N}{c}$ are already taken care of in the binomial coefficient, so we just have to take care of the terms we multiply of the form $\binom{N}{c}$, the remaining diagrams are the ones, we know, X:

$$\rightsquigarrow \frac{1}{\cancel{N!}}\left(\frac{-i\lambda}{4!}\right)^N \left\{\frac{\cancel{N!}}{\cancel{(N-c)!}\,c!}\,\frac{\cancel{(N-c)!}}{\cancel{(N-2c)!}\,c!}\cdots\frac{\cancel{(2c)!}}{c!\,c!}\,\frac{1}{n!}\left\{\int d^4y_1 \ldots d^4y_c \langle 0|T\{\phi^4(y_1)\ldots\phi^4(y_c)\}|0\rangle\right.\right.$$

$$= \frac{1}{n!} \left(\frac{-i\lambda}{4!}\right)^N \frac{1}{(c!)^n} \left(\int d^4y_1 \ldots d^4y_c \langle 0|T\{\phi^4(y_1) \ldots \phi^4(y_c)\}|0\rangle\right)^n$$

$$\overset{N=nc}{=} \frac{1}{n!} \left(\frac{-i\lambda}{4!}\right)^{nc} \frac{1}{(c!)^n} \left(\int d^4y_1 \ldots d^4y_c \langle 0|T\{\phi^4(y_1) \ldots \phi^4(y_c)\}|0\rangle\right)^n$$

$$= \frac{1}{n!} \left(\frac{1}{c!} \left(\frac{-i\lambda}{4!}\right)^c \int d^4y_1 \ldots d^4y_c \langle 0|T\{\phi^4(y_1) \ldots \phi^4(y_c)\}|0\rangle\right)^n$$

$$= \frac{1}{n!} X^n$$

Now that those m subdiagrams we want to calculate the amplitude for, consist of n_i times the <u>same</u> subdiagram X_i, therefore all others but <u>those</u> n_i subdiagrams look <u>different</u>, meaning especially that we can <u>not</u> permute between those, we get:

$$\prod_{i=1}^{m} (X_i)^{n_i} \frac{1}{n_i!} = A \leftarrow \text{total Amplitude}$$

because one can simply multiply the amplitudes of distinguishable diagrams!