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26.06.2017 Quantum Field theory 8th Exercise sheet Marvin Danke

H8) a) $\mathcal{L} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - \frac{1}{2}m^2\left(\sum_{i=1}^{3}\phi_i^2\right) - \frac{\lambda}{8}\left(\sum_{i=1}^{3}\phi_i^2\right)^2$

Now, free field, i.e. $\lambda = 0$:

$$\mathcal{L}_{free} = \sum_{i=1}^{3} \frac{1}{2}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - \frac{1}{2}m^2\left(\sum_{i=1}^{3}\phi_i^2\right)$$

which obviously represents the sum of three 1-dim KG-Lagrangians.

Prove it without using Wick's theorem, which we didn't prove for $T\phi_i\phi_j$? Or smarter way to see?

we did.

The Fourier Decomposition of the fields looks as follows:

$$\left.\begin{aligned}
&\phi_i(x) = \phi_i^+(x) + \phi_i^-(x)\\
&\phi_i^+(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\, a_k^i e^{-ikx}, \quad \phi_i^-(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_k}}\, a_k^{i\dagger} e^{ikx}\\
&\text{with } [a_k^i, a_{k'}^{j\dagger}] = \delta_{ij}(2\pi)^3\delta^{(3)}(\underline{k}-\underline{k}')\\
&[a_k^j, a_{k'}^j] = 0 = [a_k^{i\dagger}, a_{k'}^{j\dagger}]
\end{aligned}\right\} (*)$$

We now calculate:

$$T(\phi_i(x)\phi_j(y)) = T\left((\phi_i^+(x)+\phi_i^-(x))(\phi_j^+(y)+\phi_j^-(y))\right)$$

$$\overset{!\, x^0>y^0}{=} \phi_i^+(x)\phi_j^+(y) + \phi_i^+(x)\phi_j^-(y) + \phi_i^-(x)\phi_j^+(y) + \phi_i^-(x)\phi_j^-(y) \underbrace{+\phi_j^-(y)\phi_i^+(x) - \phi_j^-(y)\phi_i^+(x)}_{=0}$$

$$= \phi_i^+(x)\phi_j^+(y) + \phi_j^-(y)\phi_i^+(x) + \phi_i^-(x)\phi_j^+(y) + \phi_i^-(x)\phi_j^-(y) + [\phi_i^+(x), \phi_j^-(y)]$$

$$= :\phi_i\phi_j: + [\phi_i^+(x), \phi_j^-(y)] = :\phi_i\phi_j: + \delta_{ij}D(x-y)$$

Analogue

$$T(\phi_i(x)\phi_j(y)) \overset{y^0>x^0}{=} :\phi_i\phi_j: + \delta_{ij}D(y-x)$$

$$\leadsto T(\phi_i(x)\phi_j(y)) = :\phi_i\phi_j: + \delta_{ij}\underbrace{\{\theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x)\}}_{=: D_F(x-y) \text{ (a complex number)}}$$

$\langle 0|:\_\_:|0\rangle = 0$

$$\leadsto \langle 0|T(\phi_i(x)\phi_j(y))|0\rangle = \delta_{ij}D_F(x-y)\langle 0|0\rangle = \delta_{ij}D_F(x-y)$$

For the Feynman rules, we first take a look at the interaction term, $\mathcal{L}_{int} = -\frac{\lambda}{8}\left(\sum_{i=1}^{3}\phi_i^2\right)^2 = -\frac{\lambda}{8}(\phi_1^2+\phi_2^2+\phi_3^2)^2$

$$= -\frac{\lambda}{8}(\phi_1^4+\phi_2^4+\phi_3^4+2\phi_1^2\phi_2^2+2\phi_1^2\phi_3^2+2\phi_2^2\phi_3^2)$$

$$= -\frac{\lambda}{8}(\phi_1^4+\phi_2^4+\phi_3^4) - \frac{\lambda}{4}(\phi_1^2\phi_2^2+\phi_1^2\phi_3^2+\phi_2^2\phi_3^2)$$

From what we have proven about the Feynman propagator for this theory, it is obvious that propagators take the following form: (momentum space)

Is it momentum space? Yes

$$\phi_i \text{———} \phi_j \qquad \delta_{ij}\frac{i}{p^2-m^2+i\epsilon}$$

The next objects we want to look at are vertrees. For this we need the first order perturbation theory of

What is it actually? S-matrix element

$$\langle out|T\exp(-i\int d^4x\, \mathcal{H}_{int})|in\rangle, \quad \mathcal{H}_{int} = -\mathcal{L}_{int}$$

$S = 1 + i\mathcal{M}(2\pi)^4\delta^{(4)}(\ )$

1st order

$$= \langle out|\frac{-i\lambda}{8}\int d^4y\,(\phi_1^4+\phi_2^4+\phi_3^4+2\phi_1^2\phi_2^2+2\phi_1^2\phi_3^2+2\phi_2^2\phi_3^2)|in\rangle$$

0th order also $\neq 0$ for $|in\rangle = |out\rangle$ not interested

$$= -\frac{i\lambda}{8}\int d^4y\,\langle out|:\phi_1^4+\phi_2^4+\phi_3^4+2\phi_1^2\phi_2^2+2\phi_1^2\phi_3^2+2\phi_2^2\phi_3^2:$$
$$+ :\text{Contractions}:|in\rangle$$

What do those contractions of terms look like? just split up

for the vertex,

$$\underset{\text{we want to look at fully contracted parts only (with ext. states)}}{\Downarrow} \quad -\frac{i\lambda}{8}\int d^4y\,\langle out|:\phi_1^4+\phi_2^4+\phi_3^4+2\phi_1^2\phi_2^2+2\phi_1^2\phi_3^2+2\phi_2^2\phi_3^2:|in\rangle \quad (*)$$

This has 3 possible contributions, say 3 cases for $|in\rangle$, $|out\rangle$, which contribute to the vertex factor:

$$\text{a)}\ \langle\phi_i\phi_i|\,\phi_i(y)\phi_i(y)\phi_i(y)\phi_i(y)\,|\phi_i\phi_i\rangle \ \times 4!$$

, as first $\phi_i(y)$ has 4 possible ϕ_i's to contract with, second has 3 and so on.

Still normal ordered? → results in the contractions

◦2 $\langle \phi_i \phi_j | \phi_i(y) \phi_i(y) \phi_j(y) \phi_j(y) | \phi_i \phi_j \rangle \times 4$ as first $\phi_i(y)$ has 2 possible ϕ_i's to contract with, other $\phi_i(y)$ fixed; for $\phi_j(y)$ same

◦3 $\langle \phi_i \phi_i | \phi_i(y) \phi_i(y) \phi_j(y) \phi_j(y) | \phi_j \phi_j \rangle \times 4$ as first $\phi_i(y)$ has 2 possibilities, second $\phi_i(y)$ fixed for $\phi_j(y)$ analogue

What about interchanging ϕ_i's and ϕ_j's? Can commute those again

All together, we thus get a vertex factor of

$$-i\lambda\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

for an incoming state $|\phi_i \phi_j\rangle$ and an outgoing state $\langle \phi_k \phi_l|$

Is this the general vertex factor now? → Yes!

We calculate the outgoing/incoming external states as follows:

$$\langle 0| \phi_{(i)}(x) |p_\phi\rangle = \langle 0| \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_k e^{-ikx} + \underbrace{a_k^\dagger e^{ikx}\right)\sqrt{2E_p}\, a_p^\dagger |0\rangle}_{=0}$$

assuming p_ϕ represents same field component ϕ_i

$$= \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{E_p}{E_k}}\, e^{-ikx} \langle 0| \underbrace{a_k a_p^\dagger}_{= a_p^\dagger a_k + [a_k, a_p^\dagger]} |0\rangle$$

$a_p^\dagger a_k$: $=0$

$$= \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{E_p}{E_k}}\, e^{-ikx} \langle 0| (2\pi)^3 \delta^{(3)}(k-p) |0\rangle$$

$$= e^{-ipx} \quad (p\cdot x \text{ is a } \underline{\text{Lorentz}}\ \underline{\text{(4-dim.)}} \text{ scalar})$$

As we will still integrate over x in the taylor series of the S-Matrix, those exponential functions will turn into a factor of 1.

Will those really turn into a 1 in momentum space?

Let's go back to (*) and calculate the scattering amplitude $\phi_i \phi_j \to \phi_k \phi_l$ to lowest order in λ:

$$\therefore -\frac{i\lambda}{8} \int d^4y \, \langle \phi_k \phi_l | : \phi_1^4(y) + \phi_2^4(y) + \phi_3^4(y) + 2\phi_1^2(y)\phi_2^2(y) + 2\phi_1^2(y)\phi_3^2(y) + 2\phi_2^2(y)\phi_3^2(y) | \phi_i \phi_j \rangle$$

$$= -i\lambda \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) \left\{ (2\pi)^4 \delta^{(4)}(p_k + p_l - p_i - p_j) \right\}$$

Where the last factor comes from applying the contraction of the ϕ_i's / ϕ_j's 4 times (see external states calculation) and thus receiving:

$$\int d^4y \, e^{iy(p_k + p_l - p_i - p_j)} = (2\pi)^4 \delta^{(4)}(p_k + p_l - p_i - p_j)$$

Calculate again or just use Feynman rules? → kind of what we just calculate

Where from $(2\pi)^4$ with Feynman rules? → taken care of in $iM(2\pi)^4\delta^{(4)}$

$k = l = i = j$ ⇒ $\delta^{(4)}(0)$?

→ Notation: ϕ_i has ≙ mom. p_1, $\phi_j \hat{=} p_2$ etc

What for $\mathcal{L}_0$ given?

b) $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\partial_\mu \psi)(\partial^\mu \psi) - \frac{1}{2}\mu^2\psi^2$$
$$+ \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \frac{1}{2}M^2\chi^2$$

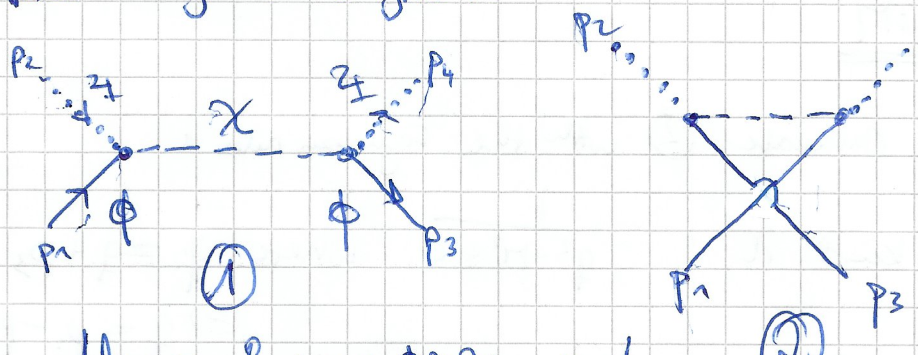
$$\mathcal{L}_1 = -g\phi\psi\chi$$

We are looking at the process $\phi(p_1) + \psi(p_2) \to \phi(p_3) + \psi(p_4)$ for which we get vanishing 0^{th} and 1^{st} order in perturbation theory of the S-Matrix, as $\mathcal{L}_1$ contains an uneven number of fields in 1^{st} order and $|in\rangle$, $\langle out|$ contain an even number $\Rightarrow$ cannot be fully contracted. 0^{th} order vanishes as it represents no scattering.

0^{th} order possible if $|in\rangle = |out\rangle$? $\Rightarrow$ 1^{th} not possible as uneven – even states

We will look at the $\mathcal{O}(g^2)$ contribution for which we get 2 different Feynman diagrams:

How to see fast that those are the only 2 Feynman diagrams? $\Rightarrow$ Üben; durchrechnen!



with $\mathcal{H}_{int} = -\mathcal{L}_{int} = g\phi\psi\chi$ and

$$S = \langle \phi_3 \psi_4 | T\exp(-i\int d^4y\, \mathcal{H}_{int}) | \phi_1 \psi_2 \rangle$$

$$\overset{\mathcal{O}(g^2)}{\approx} \langle \phi_3 \psi_4 | \frac{(-ig)^2}{2} T \int d^4x\, d^4y\, \phi(x)\psi(x)\chi(x)\phi(y)\psi(y)\chi(y) | \phi_1 \psi_2 \rangle =: S^{(2)}$$

$$\overset{\text{Wick}}{=} -\frac{g^2}{2}\int d^4x\, d^4y \langle \phi_3 \psi_4 | :\phi(x)\psi(x)\chi(x)\phi(y)\psi(y)\chi(y): | \phi_1 \psi_2 \rangle$$
$$+ \text{: contractions :}$$

Does one simply add the other permutations/contractions? $\Rightarrow$ Yes, actually it's twice the diagram e.g. with x, y exchanged!

For fixed x, y, one can see now that 1 and 2 are the only possible diagrams, as $\chi(x)$ and $\chi(y)$ have to be contracted with each other (no external χ fields), leaving to possibilities for ϕ_1 to contact with same (vertex) $\phi(x)$ / $\phi(y)$ as ψ_2 or ($\psi(x)$ / $\psi(y)$) with same vertex as ψ_4

Now, we just one to calculate the contribution of ①, yielding:

$$\tilde{S}^{(2)} = -\frac{g^2}{2}\int d^4x\, d^4y \,\langle \phi_3 \psi_4 |\, \chi(x)\chi(y)\, \phi(x)\psi(x)\phi(y)\psi(y)\,|\phi_1 \psi_2\rangle$$

2 possible ways: contract $\phi(x)$ with ϕ_1 or ϕ_3, other: $\phi(y)$ and $\psi(x)$, $\psi(y)$ fixed (for given diagram ①): say x,y inter-changeable

$$= -g^2 \int d^4x\, d^4y\, \chi(x)\chi(y)\, e^{-ix(p_1+p_2)}\, e^{iy(p_3+p_4)}$$

can we just use the propagator?

$$= -g^2 \int d^4x\, d^4y \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} e^{-ip(x-y)} e^{-ix(p_1+p_2)} e^{iy(p_3+p_4)}$$

$$\overset{p \leftrightarrow -p}{=} -g^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} \int d^4x\, e^{+ix(p-p_1-p_2)} \int d^4y\, e^{-iy(p-p_3-p_4)}$$

$D(x-y) = D(y-x)$? (using $p \leftrightarrow -p$)

$$= -g^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - M^2 + i\epsilon} (2\pi)^8 \delta^{(4)}(p-p_1-p_2)\, \delta^{(4)}(p-p_3-p_4)\,.$$

use 1 δ-distr. for $\int d^4p$, & substitute the other one

$$= -g^2 (2\pi)^4 \frac{i}{s - M^2 + i\epsilon} \delta^{(4)}(p_1+p_2-p_3-p_4)\,, \quad s = (p_1+p_2)^2$$

$$\overset{\epsilon \to 0}{=} \underbrace{\frac{-g^2}{s - M^2}}_{=:\mathcal{M}} \left\{ i (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4) \right\}$$

✓

We want to calculate ② with the Feynman rules:

• Propagator of kind χ: $\frac{i}{q^2 - M^2 + i\epsilon}$ where $p_2 = q + p_3$

$\Rightarrow q = p_2 - p_3 = \sqrt{u}$

$\left(p_1 + q = p_4 \quad \delta(q - (p_2 - p_3))\, \delta(q - (p_4 - p_1))\, \delta(p_4 - p_1 - p_2 + p_3) \right)$

• 2 vertices: $(-ig)^2$

(• momentum conservation: $(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$)

$$\rightsquigarrow i\mathcal{M} = -i \frac{g^2}{u - M^2}$$

✓ ✓

No $(2\pi)^4$ in Feynman rules? → it's in $iT = i\mathcal{M}(2\pi)^4\delta^{(4)}$

Do not have to calculate a symmetry factor? → only if contracted inside (exchangeable)