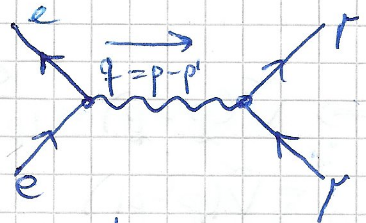
Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

06.07.2017 Quantum Field Theory 9th Exercise Marvin Zanke

A10)

a) 

$$e^s(p) + \mu^r(k) \longrightarrow e^{s'}(p') + \mu^{r'}(k')$$

massive, mass m_μ

For fermion/antifermion → also e^+, μ^+ propagator possible? → not for ferm. antiferm. scattering

$$i\mathcal{M} = \bar{u}_e^{s'}(p')(-ie\gamma^\mu)u_e^s(p)\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\bar{u}_\mu^{r'}(k')(-ie\gamma^\nu)u_\mu^r(k)$$

$$= ie^2\,\bar{u}_e^{s'}(p')\gamma^\mu u_e^s(p)\frac{g_{\mu\nu}}{(p-p')^2+i\epsilon}\bar{u}_\mu^{r'}(k')\gamma^\nu u_\mu^r(k)$$

$$\bar{u}_e^{s'}(p')\gamma^\mu u_e^s(p) = \begin{cases} u_e^{\dagger s'}(p')\underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}u_e^s(p) = u_e^{\dagger s'}(p')u_e^s(p), & \mu=0 \\ u_e^{\dagger s'}(p')\gamma^0\gamma^i u_e^s(p), & \mu=i \end{cases}$$

$\gamma^0 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$, $\gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}$

$u^s(p) = \sqrt{2m}\begin{pmatrix}\xi^s\\ 0\end{pmatrix}$ in non-rel. limit

No $\mathcal{O}(-)$ diagram? → only $e\gamma \to e^-$, $\gamma \to e^-e^+$ etc. …

$u^s(p) = \sqrt{E_p+m}\begin{pmatrix}\xi^s\\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\xi^s\end{pmatrix}$, $\frac{\vec{p}}{2m} \to 0$

$$= \begin{cases} 2m_e\,\delta_{ss'}, & \mu=0 \\ 2m_e\,(\xi^{s\dagger}, 0)\begin{pmatrix}0\\ -\sigma^i\xi^s\end{pmatrix} = 0, & \mu=i \end{cases}$$

$\bar{u}_\mu^{r'}(k')\gamma^\nu u_\mu^r(k)$ analogue

$$= ie^2\,(2m_e\delta_{ss'})\frac{g_{00}}{(p-p')^2+i\epsilon}(2m_\mu\delta_{rr'})$$

Hier oben benötigt man in Peskin $p\to 0$? wofür?

$$= \frac{ie^2}{(p-p')^2+i\epsilon}2m_e\,2m_\mu\,\delta_{ss'}\delta_{rr'}$$

$$(p-p')^2 = \underbrace{(E_p-E_{p'})^2}_{} - |\vec{p}-\vec{p}'|^2 = -|\vec{p}-\vec{p}'|^2 + \mathcal{O}(p^4) + \mathcal{O}(p'^4)$$

$$= \left(\sqrt{p^2+m_e^2} - \sqrt{p'^2+m_e^2}\right)^2 = (m_e - m_e + \mathcal{O}(p^2) + \mathcal{O}(p'^2))^2$$

$$\approx \frac{-ie^2}{|\vec{p}-\vec{p}'|^2 - i\epsilon}2m_e\,2m_\mu\,\delta_{ss'}\delta_{rr'}$$

$\epsilon \to 0$ hier? → ja, (nur 2?) bei loops benötigt

$$\xrightarrow{\epsilon\to 0} \frac{-ie^2}{|\vec{p}-\vec{p}'|^2}2m_e\,2m_\mu\,\delta_{ss'}\delta_{rr'}$$

b) $iM = -i\tilde V(p'-p)\, \underbrace{2m_\mu \delta_{rr'}\, 2m_e \delta_{ss'}}$

$\overset{!}{=} \frac{-ie^2}{|p-p'|^2} \underbrace{2m_e\delta_{ss'}\, 2m_\mu\delta_{rr'}} \iff \tilde V(p'-p) = \frac{e^2}{|p-p'|^2}$

Why not $V = \int \ldots V(p'-p)$ and $\langle p'|iT|p\rangle$ given like this here?

$q \equiv p-p'$ $\qquad \equiv \tilde V(|p'-p|)$

$V(\underline{x}) = \frac{1}{(2\pi)^3}\int d^3q\, V(\underline{q})\, e^{-i\underline{q}\,\underline{x}} = \frac{1}{(2\pi)^3}\, 2\pi \int_0^\infty dq \int_{-1}^{1} d\cos\theta\; e^2\, e^{-iq|x|\cos\theta}$

$q = p-p'$ ok? → just definition ✓ — integration over all

$r \equiv |x|$: $= \frac{e^2}{4\pi^2}\int_0^\infty dq \left\{\frac{1}{-iqr}\, e^{-iqr\cos\theta}\right\}\Big|_{-1}^{1}$

$= \frac{-e^2}{4i\pi^2}\int_0^\infty dq\, \frac{e^{-iqr}-e^{iqr}}{qr} \overset{\sin x = \frac{1}{2i}(e^{ix}-e^{-ix})}{=} \frac{+e^2}{2\pi^2}\int_0^\infty dq\, \frac{\sin(qr)}{qr}$

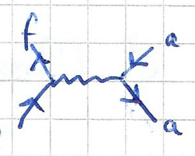
$= \frac{e^2}{2\pi^2 r}\underbrace{\int_0^\infty dz\, \frac{\sin(z)}{z}}_{=\pi/2,\ \text{Mathematica, alternatively via residue theorem}} = \frac{e^2}{4\pi r}$ $\quad \xrightarrow{r\to\infty} 0,\ \xrightarrow{r\to 0} \infty$ $\Big\}$ repulsive ✓

$\tilde V = \frac{e^2}{|p-p'|^2 - i\epsilon} \ldots$

c) $H_{int} = \int d^3x\; e\bar\psi\gamma^\mu\psi A_\mu$

For fermion-anti-fermion scattering, we see that the contractions

$iM \sim \langle p'_f k'_a | \int d^4x\, d^4y\; \bar\psi_x\gamma^\mu\psi_x A_{\mu x}\, \bar\psi_y \gamma^\nu \psi_y A_{\nu y} | p_f k_a\rangle$



$\sim \langle 0| a_{k'}\, a_{p'}\, \bar\psi_x\, \psi_x\, \bar\psi_y\, \psi_y\, a^\dagger_{p}\, a^\dagger_{k}|0\rangle$ need ψ_x to be swapped to the left twice, $\bar\psi_y$ to be swapped to the left once after that to untangle them ⇒ (−1) sign

anti-fermion–anti-fermion: ψ_x left 1×, $\bar\psi_y$ left 2×, ψ_x right 1× $\Big\}$ (+1)

Additionally, we have one part of iM looking like $\bar v^r(k)\gamma^\nu v^{r'}(k')$

and as $\bar v^r(k)\gamma^i v^{r'}(k') = v^{\dagger r}(k)\gamma^0\gamma^i v^{r'}(k') = 2m\,(0, -\xi^{\dagger r})\begin{pmatrix}\sigma^i\xi^{r'}\\ 0\end{pmatrix} = 0$

$\boxed{v^r(k) \approx \sqrt{2m}\begin{pmatrix}0\\ \xi^r\end{pmatrix}}$ in n.r.

we only get the $\bar v^r(k)\gamma^0 v^{r'}(k')$

$= v^{\dagger r}(k) v^{r'}(k') = 2m\,\delta_{rr'}$ contribution and therefore a

total (−1) sign ✓ ⇒ potential attractive ✓

Is $e^- p^+ \to e^- p^+$ possible? or only $e^- e^+ \to e^- e^+$ → $p\bar p$

For anti fermion–antifermion scattering, we would twice have factors of $\bar v^r(k)\gamma^0 v^{r'}(k')$ ⇒ no sign; and we would get no sign for the contractions, just like in the fermion–fermion case ⇒ (+1) sign ⇒ repulsive potential (see above)

Is (+1) for $ff \to ff$ clear in general?

Consistent with alike charged particles repulsing and unlike attracting each other ✓