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String Theory Exercise 10 Homework

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13.12.2018 1.1.

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already N.O.

$$T(z)\,\psi^\mu(w) = \{ -\tfrac{1}{2} :\partial X^\mu(z)\partial X_\mu(z): - \tfrac{1}{2} :\psi^\nu(z)\partial\psi_\nu(z): \}\psi^\mu$$

terms like $\langle \partial X \psi \rangle$ vanish, as the Correlator / propagator is zero. (Wick's theorem)

2d N=1 S.S.? ⇝ 2dim S.S w/one super-charge (Q from TFP)

SUGRA ⊃ SCFT?

$$= -\tfrac{1}{2} :\psi^\nu(z)\partial\psi_\nu(z): \psi^\mu(w)$$

$$= -\tfrac{1}{2}\{ \text{N.O.} - \langle \psi^\nu(z)\psi^\mu(w)\rangle :\partial\psi_\nu(z): + \langle \partial\psi^\nu(z)\psi^\mu(w)\rangle :\psi_\nu(z): \}$$

Commute $\partial\psi$'s

why $\alpha' = 2$?

$$= -\tfrac{1}{2}\{ \text{N.O.} - \eta^{\mu\nu}\frac{1}{z-w} :\partial\psi_\nu(z): - \eta^{\mu\nu}\frac{1}{(z-w)^2} :\psi_\nu(z): \}$$

WS super-Currents?

$$= \frac{1/2}{(z-w)^2}\psi^\mu(z) + \frac{1/2}{z-w}\partial\psi^\mu(z) + \text{N.O.}$$

$$= \frac{1/2}{(z-w)^2}\{\psi^\mu(w) + \partial_w\psi^\mu(w)(z-w) + \text{reg.}\} + \frac{1/2}{z-w}\{\partial\psi^\mu(w) + \text{reg.}\} + \text{N.O.}$$

why the $\partial\psi^\mu$ not der. with respect to w here?

N.O. also regular

$$= \frac{1/2}{(z-w)^2}\cancel{\psi^\mu(w)} + \frac{\partial_w\psi^\mu(w)}{z-w} + \text{reg.}$$

As $T(z)$ only contains fields X and ψ, we find $(h,\bar{h}) = (\tfrac{1}{2}, 0)$

as $\bar{T}(\bar{z})\psi^\mu(w) = 0$

But the derivative term should still be there? ✓

⇝ Is there but vanishes, as $\frac{\bar{\partial}_w\psi^\mu(w)}{\bar{z}-\bar{w}} = 0$

Now

$$T(z)\,T_F(w) = -\frac{i}{4}\left\{ :\partial X^\mu(z)\partial X_\mu(z): + :\psi^\mu(z)\partial\psi_\mu(z): \right\} \times \left\{ :\psi^\nu(w)\partial X_\nu(w): \right\}$$

$$= -\frac{i}{4}\Big\{ \text{N.O.} + \cancel{\langle \partial X^\mu(z)\psi^\nu(w)\rangle}\langle \partial X_\mu(z)\partial X_\nu(w)\rangle$$
$$+ \langle \partial X^\mu(z)\partial X^\nu(w)\rangle \cancel{\langle \partial X_\mu(z)\psi_\nu(w)\rangle}$$
$$+ \cancel{\langle \partial X^\mu(z)\psi^\nu(w)\rangle} :\partial X_\mu(z)\partial X_\nu(w):$$
$$+ \langle \partial X_\mu(z)\partial X_\nu(w)\rangle :\partial X^\mu(z)\psi^\nu(w):$$
$$+ \langle \partial X^\mu(z)\partial X^\nu(w)\rangle :\partial X_\mu(z)\psi_\nu(w):$$
$$+ \cancel{\langle \partial X_\mu(z)\psi_\nu(w)\rangle} :\partial X^\mu(z)\partial X^\nu(w):$$

$$- \langle \psi^\mu(z)\psi^\nu(w)\rangle \cancel{\langle \partial\psi_\mu(z)\partial X_\nu(w)\rangle}$$
$$+ \cancel{\langle \psi^\mu(z)\partial X^\nu(w)\rangle}\langle \partial\psi_\mu(z)\psi_\nu(w)\rangle$$
$$- \langle \psi^\mu(z)\psi^\nu(w)\rangle :\partial\psi_\mu(z)\partial X_\nu(w):$$

~~an~~ commute $\psi^\nu(w)$ w/ $\partial X_\nu(w)$ first

$$+ \cancel{\langle \partial\psi_\mu(z)\partial X_\nu(z)\rangle} :\psi^\mu(z)\psi^\nu(w):$$
$$+ \cancel{\langle \psi^\mu(z)\partial X^\nu(w)\rangle} :\partial\psi_\mu(z)\psi_\nu(w):$$
$$+ \langle \partial\psi_\mu(z)\psi_\nu(w)\rangle :\psi^\mu(z)\partial X^\nu(w):$$

Also need to anticommute them if take only one prop. and the other still N.O.?

$$= -\frac{i}{4}\Big\{ -\eta^{\mu\nu}\frac{2}{(z-w)^2} :\partial X_\mu(z)\psi_\nu(w): - \eta^{\mu\nu}\frac{1}{z-w} :\partial\psi_\mu\,\partial X_\nu(w):$$
$$- \eta^{\mu\nu}\frac{1}{(z-w)^2} :\psi_\mu(z)\partial X_\nu(w): \Big\}$$

$$= \frac{i}{4}\eta^{\mu\nu}\Big\{ \frac{2}{(z-w)^2}\left(\partial X_\mu(w)\psi_\nu(w) + \partial^2 X_\mu(w)\psi_\nu(w)(z-w) + \text{reg.}\right)$$
$$+ \frac{1}{z-w}\left(\partial\psi_\mu(w)\partial X_\nu(w) + \text{reg.}\right)$$
$$+ \frac{1}{(z-w)^2}\left(\psi_\mu(w)\partial X_\nu(w) + \partial\psi_\mu(w)\partial X_\nu(w)(z-w) + \text{reg.}\right)\Big\}$$

$$= \frac{i}{4}\eta^{\mu\nu}\Big\{ \frac{1}{(z-w)^2}\left(2\partial X_\mu(w)\psi_\nu(w) + \psi_\mu(w)\partial X_\nu(w)\right)$$
$$+ \frac{1}{(z-w)}\left(2\partial^2 X_\mu(w)\psi_\nu(w) + \partial\psi_\mu(w)\partial X_\nu(w) + \partial\psi_\mu(w)\partial X_\nu(w)\right) \text{reg}$$

$$= \frac{3/2}{(z-w)^2}\cancel{T_F(w)} + \frac{\partial_w T_F(w)}{z-w} + \text{reg.}$$

1.2:

$$\delta_\epsilon X^\mu(z) = -[T_{F_\epsilon}, X^\mu(z)] = -\oint_{C_0} \frac{dw}{2\pi i}[\epsilon(w)T_F(w), X^\mu(z)]$$

Connection between conf. and superconf. trafos?

$$= \oint_{C_w} \frac{dw}{2\pi i} R(\epsilon(w)T_F(w)X^\mu(z))$$

$$T_F(w)X^\nu(z) = \frac{i}{2} :\psi^\mu(w)\partial X_\mu(w): X^\nu(z)$$

$$= \frac{i}{2}\{\langle\partial X^\mu(w)X^\nu(z)\rangle\psi_\mu(w)\}$$

$$= -\frac{i}{2}\eta^{\mu\nu}\frac{1}{w-z}\psi_\mu(w)$$

$$= \frac{i}{2}\oint_{C_w}\frac{dw}{2\pi i}\epsilon(w)\frac{1}{w-z}\psi^\mu(w)$$

Cauchy-Riemann formula

$$= \frac{i}{2}\epsilon(z)\psi^\mu(z)$$

$$\delta_\epsilon \psi^\mu(z) = -[T_{F_\epsilon}, \psi^\mu(z)] = -\oint_{C_0}\frac{dw}{2\pi i}[\epsilon(w)T_F(w), \psi^\mu(z)]$$

No $\bar{T}_F$ in δ_ϵ? For $\delta_{\bar\epsilon}$ then?

$$= -\oint_{C_w}\frac{dw}{2\pi i}R(\epsilon(w)T_F(w)\psi^\mu(z))$$

$$T_F(w)\psi^\mu(z) = \frac{i}{2}:\psi^\nu(w)\partial X_\nu(w):\psi^\mu(z)$$

$$= \frac{i}{2}\{\langle\psi^\nu(w)\psi^\mu(z)\rangle\partial X_\nu(w)\}$$

$$= \frac{i}{2}\eta^{\mu\nu}\frac{1}{w-z}\partial X_\nu(w)$$

$$= -\frac{i}{2}\oint_{C_w}\frac{dw}{2\pi i}\epsilon(w)\frac{1}{w-z}\partial X^\mu(w)$$

$$= -\frac{i}{2}\epsilon(z)\partial X^\mu(z)$$

1.3.

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] F(z) = \delta_{\epsilon_1}\delta_{\epsilon_2} F(z) - \delta_{\epsilon_2}\delta_{\epsilon_1} F(z)$$

$$= -\delta_{\epsilon_1}[T_{F\epsilon_2}, F(z)] + \delta_{\epsilon_2}[T_{F\epsilon_1}, F(z)]$$

$$= [T_{F\epsilon_1}[T_{F\epsilon_2}, F(z)]] - [T_{F\epsilon_2}, [T_{F\epsilon_1}, F(z)]]$$

Cyclicity

$$\overset{\downarrow}{=} -[F(z), [T_{F\epsilon_1}, T_{F\epsilon_2}]] = [[T_{F\epsilon_1}, T_{F\epsilon_2}], F(z)]$$

$$= \oint_{C_0}\oint_{C_0} \frac{dw}{2\pi i}\frac{dy}{2\pi i} [[\epsilon_1(w) T_F(w), \epsilon_2(y) T_F(y)], F(z)]$$

additional (-1) because ϵ and T_F Grassmannian

$$= \left[\oint_{C_0} \frac{dy}{2\pi i} \oint_{C_y} \frac{dw}{2\pi i} \epsilon_1(w) \epsilon_2(y) R(T_F(w) T_F(y)), F(z)\right]$$

see (1.11) (proven later)

$$\overset{\downarrow}{=} \left[\oint_{C_0} \frac{dy}{2\pi i} \oint_{C_y} \frac{dw}{2\pi i} \epsilon_1(w) \epsilon_2(y) \left\{ \frac{c/6}{(w-y)^3} + \frac{1/2}{(w-y)} T(w) \right\}, F(z)\right]$$

Wrong result in (1.11)? → correct like this

C.I.

$$\overset{\downarrow}{=} \left[\oint_{C_0} \frac{dy}{2\pi i} \left\{ \frac{c}{12} \partial^2\epsilon_1(y) \epsilon_2(y) + \frac{1}{2} \epsilon_1(y)\epsilon_2(y) T(y) \right\}, F(z)\right]$$

2 Should be a minus? → yes, see radial ordering above.

$[\epsilon\epsilon, F] = 0$

$$\overset{\downarrow}{=} \left[\oint_{C_0} \frac{dy}{2\pi i} \frac{1}{2} \epsilon_1(y) \epsilon_2(y) T(y), F(z)\right]$$

$$= [T_\xi, F(z)] \quad \text{with} \quad \xi = \tfrac{1}{2}\epsilon_1\epsilon_2$$

expect minus, maybe $\xi = \frac{1}{2}\epsilon_2\epsilon_1$ or $\xi = -\frac{1}{2}\epsilon_1\epsilon_2$
or see above

1.4

$$T(z)T(w) = \left(T_b(z) + T_f(z)\right)\left(T_b(w) + T_f(w)\right)$$

bosons and fermions do not mix (cor. vanish) $\rightarrow$

$$= T_b(z)T_b(w) + T_f(z)T_f(w)$$

$$= \frac{\frac{1}{2}D}{(z-w)^4} + \frac{2T_b(w)}{(z-w)^2} + \frac{\partial T_b(w)}{z-w}$$

$$+ \frac{1/4\, D}{(z-w)^4} + \frac{2T_f(w)}{(z-w)^2} + \frac{\partial T_f(w)}{z-w} + \text{reg}$$

And what about conf. w/ conf.?

additional factor of D because of the metric from components of $X^\mu \psi^\mu$ (only for terms where 2 prop. appear w/ $(z-w)^4$

$$= \frac{3/4\, D}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

$T(z)T_F(w)$ already calculated in 1.1

$$T_F(z)T_F(w) = -\frac{1}{4} :\psi^\mu(z)\partial X_\mu(z)::\psi^\nu(w)\partial X_\nu(w):$$

No commutating necessary?

$$= -\frac{1}{4}\{N.O. + \langle\psi^\mu(z)\psi^\nu(w)\rangle\langle\partial X_\mu(z)\partial X_\nu(w)\rangle$$
$$+ \cancel{\langle\psi^\mu(z)\partial X^\nu(w)\rangle}\langle\partial X_\mu(z)\psi_\nu(w)\rangle$$
$$+ \langle\psi^\mu(z)\psi^\nu(w)\rangle :\partial X_\mu(z)\partial X_\nu(w):$$
$$+ \langle\partial X_\mu(z)\partial X_\nu(w)\rangle :\psi^\mu(z)\psi^\nu(w):$$
$$+ \cancel{\langle\psi^\mu(z)\partial X^\nu(w)\rangle} :\partial X_\mu(z)\psi_\nu(w):$$
$$+ \cancel{\langle\partial X_\mu(z)\psi_\nu(w)\rangle} :\psi^\mu(z)\partial X^\nu(w):\}$$

$$= -\frac{1}{4}\{N.O. - \eta^{\mu\nu}\frac{1}{z-w}\cdot\eta_{\mu\nu}\frac{1}{(z-w)^2} + \eta^{\mu\nu}\frac{1}{z-w}:\partial X_\mu(z)\partial X_\nu(w):$$
$$- \eta^{\mu\nu}\frac{1}{(z-w)^2}:\psi^\mu(z)\psi^\nu(w):\}$$

$$= -\frac{1}{4}\{N.O. - \frac{D}{(z-w)^3} + \frac{1}{(z-w)}\{\partial X_\mu(w)\partial X^\mu(w) + \text{reg.}\}$$
$$- \frac{1}{(z-w)^2}\{\psi^\mu(w)\psi_\mu(w) + (z-w)\partial\psi_\mu(w)\psi^\mu(w) + \text{reg}\}\}$$

$$= \frac{D/4}{(z-w)^3} + \frac{1/4}{(z-w)^2}\psi_\mu(w)\psi^\mu(w) - \frac{1/4}{z-w}\partial X_\mu(w)\partial X^\mu(w)$$
$$+ \frac{1/4}{z-w}\underbrace{\partial\psi_\mu(w)\psi^\mu(w)}_{= -\psi^\mu(w)\partial\psi_\mu(w)} + \text{reg}(N.O.)$$

$$= \frac{D/4}{(z-w)^3} + \frac{1/2}{z-w} T(w)$$

$\uparrow$ $\psi_\mu \psi^\mu = 0$ as $\{\psi, \psi\} = 0$

$\psi^\mu(z+2\pi) = \psi^\mu$ etc.?

1.5 $\psi^\mu(z) = \sum_{r\in\mathbb{Z}+a} \frac{\psi^\mu_r}{z^{r+1/2}} \Rightarrow \psi^\mu_\ell = \oint \frac{dz}{2\pi i}\, \psi^\mu(z)\, z^{\ell-1/2}$

Contour C_0, i.e. around 0?

as then $\psi^\mu_\ell = \oint \frac{dz}{2\pi i} \sum_{r\in\mathbb{Z}+a} \frac{\psi^\mu_r}{z^{r+1/2}} z^{\ell-1/2} = \sum_{r\in\mathbb{Z}+a} \oint \frac{dz}{2\pi i} \frac{\psi^\mu_r}{z^{r-\ell+1}}$

$$= \sum_{r\in\mathbb{Z}+a} \psi^\mu_r \delta_{r\ell} = \psi^\mu_\ell \checkmark$$

What if $r,s = -1/2$?

$$\Rightarrow \{\psi^\mu_r, \psi^\nu_s\} = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i}\, z^{r-1/2} w^{s-1/2} R(\psi^\mu(z)\psi^\nu(w))$$

$$= \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i}\, z^{r-1/2} w^{s-1/2} \left(\frac{\eta^{\mu\nu}}{z-w} + \text{reg.}\right)$$

only the pole at w contributes as contour C_w around w

$$= \eta^{\mu\nu} \oint_{C_0} \frac{dw}{2\pi i}\, w^{r-1/2} w^{s-1/2} = \eta^{\mu\nu} \oint_{C_0} \frac{dw}{2\pi i}\, \underbrace{w^{r+s-1}}_{\frac{1}{w^{-r-s+1}}}$$

Formula also valid for $\{\_,\_\}$?

$$= \eta^{\mu\nu} \delta_{r+s,0}$$

Completely analogous for $\{\tilde\psi^\mu_r, \tilde\psi^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}$

1.6.

$[L_m, L_n]$ has already been calculated several times.

We note for
$$T_F(z) = \sum_{r\in\mathbb{Z}+a} \frac{G_r}{z^{r+3/2}}$$

$$\rightsquigarrow \quad G_\ell = \oint_{C_0} \frac{dz}{2\pi i} z^{\ell+1/2} T_F(z)$$

and for
$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$$

$$\rightsquigarrow \quad L_m = \oint_{C_0} \frac{dz}{2\pi i} z^{\ell+1} T(z)$$

Then

$$\{G_r, G_s\} = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{r+1/2} w^{s+1/2} R(T_F(z) T_F(w))$$

Why now the other expansion w/o factor of 1/4?

other exp. of $T_F T_F$ w/ the factor of 2.

$$= \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{r+1/2} w^{s+1/2} \left\{ \frac{\frac{2c}{3}}{(z-w)^3} + \frac{2T(w)}{(z-w)} \right\}$$

$$= \oint_{C_0} \frac{dw}{2\pi i} \left\{ \frac{2c}{3} \frac{1}{2} (r+1/2)(r-1/2) w^{r-3/2} w^{s+1/2} + 2T(w) w^{r+1/2} w^{s+1/2} \right\}$$

$$= \oint_{C_0} \frac{dw}{2\pi i} \left\{ \frac{c}{3} (r^2 - 1/4) w^{r+s-1} + 2T(w) w^{r+s+1} \right.$$

$$= 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}$$

$$[L_m, G_r] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{r+1/2} R(T(z) T_F(w))$$

$$= \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{r+1/2} \left\{ \frac{3/2\, T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} \right\}$$

$$= \oint_{C_0} \frac{dw}{2\pi i} w^{r+1/2} \left\{ \frac{3}{2}(m+1) w^m T_F(w) + w^{m+1} \partial T_F(w) \right\}$$

$$= \oint \frac{dw}{2\pi i} \left\{ \frac{3}{2}(m+1) w^{m+r+1/2} T_F(w) + w^{m+r+3/2} \partial T_F(w) \right\}$$

int. by parts

$$\overset{\downarrow}{=} \frac{3}{2}(m+1)\, G_{m+r} - \oint \frac{dw}{2\pi i} T_F(w)\, (m+r+\tfrac{3}{2})\, w^{m+r+1/2}$$

does tot derivative in contour integral really vanish

$$= \frac{3}{2}(m+1)\, G_{m+r} - (m+r+\tfrac{3}{2})\, G_{m+r}$$

$$= \frac{m}{2} G_{m+r} - r\, G_{m+r}$$

$$= \frac{m-2r}{2} G_{m+r}$$