Disclaimer

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String Theory Exercises Homework 1 Marvin Zanke

$\frac{23}{24}$

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1)

1.1 $S_p = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\,(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})$

$X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu + b^\mu$ Poincaré transformation

We find $\partial_\alpha X^\mu \longmapsto \partial_\alpha(\Lambda^\mu{}_\nu X^\nu + b^\mu) = \Lambda^\mu{}_\nu \partial_\alpha X^\nu$

$\rightsquigarrow S_p = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\,(h^{\alpha\beta}\Lambda^\mu{}_\kappa \partial_\alpha X^\kappa \Lambda^\nu{}_\lambda \partial_\beta X^\lambda \eta_{\mu\nu})$

$= -\frac{T}{2}\int d^2\sigma \sqrt{-h}\,(h^{\alpha\beta}\partial_\alpha X^\kappa \partial_\beta X^\lambda \underbrace{\Lambda^\mu{}_\kappa \eta_{\mu\nu}\Lambda^\nu{}_\lambda}_{\Lambda^T\eta\Lambda = \eta}$

$= -\frac{T}{2}\int d^2\sigma \sqrt{-h}\,(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})$ ✓

why Λ indep. of σ?
$\rightsquigarrow$ just a Lorentz Trafo

$\phi^a \to \phi^a + \delta\phi^a$, $\delta\phi^a = \epsilon^i f_i{}^a(\phi_b)$ invariance of $\mathcal{L}$
(ϵ^i: infinit.)

$\longrightarrow \epsilon^i j_i^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi^a)}\delta\phi^a$ conserved current j_i^α

For Poincaré transformations: $X^\mu \mapsto X^\mu + \epsilon^\mu$ (1)

(infinitesimal variations) $X_\mu \to X_\mu + \epsilon a_{\mu\nu}X^\nu$, $a_{\mu\nu} = -a_{\nu\mu}$ (2)

$h^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\longrightarrow$ (1): $\delta X^\mu = \epsilon^\mu \rightsquigarrow \epsilon^i j_i^\gamma = \frac{\partial\mathcal{L}}{\partial(\partial_\gamma X^\kappa)}\delta X^\kappa$

$= -\frac{T}{2}\sqrt{-h}\,(h^{\gamma\beta}\partial_\beta X^\nu \eta_{\kappa\nu} + h^{\alpha\gamma}\partial_\alpha X^\mu \eta_{\mu\kappa})\epsilon^\kappa$

$\overset{h\text{ sym.}}{\underset{\eta\text{ sym.}}{=}} -T\sqrt{-h}\,h^{\gamma\beta}\partial_\beta X^\nu \eta_{\kappa\nu}\epsilon^\kappa$ ✓

$= -T\sqrt{-h}\,h^{\gamma\beta}\partial_\beta X_\kappa \epsilon^\kappa$

$\epsilon^i \neq \epsilon^\kappa$?
$\rightsquigarrow$ rename

$\rightsquigarrow j_\kappa^\gamma = -T\sqrt{-h}\,h^{\gamma\beta}\partial_\beta X_\kappa$

Inserting $h^{\alpha\beta}$: $h^{00} = -1$, $h^{11} = 1$, else $= 0$

$\rightsquigarrow j_\kappa^\gamma = -T\,\{-\delta_{\gamma 0}\partial_0 X_\kappa + \delta_{\gamma 1}\partial_1 X_\kappa\}$

② : $\delta X_\mu = \epsilon a_{\mu\nu} X^\nu$

$$\rightsquigarrow \epsilon^i j_i^\gamma = -\frac{T}{2}\sqrt{-h}\left(h^{\gamma\beta}\partial_\beta X^\nu \eta_{\kappa\nu} + h^{\alpha\gamma}\partial_\alpha X^\mu \eta_{\mu\kappa}\right)\epsilon a^{\kappa\delta} X_\delta$$

$$\rightsquigarrow j^\gamma \overset{h_{ij}}{\underset{\text{sym}}{=}} -T\sqrt{-h}\, h^{\gamma\beta}\partial_\beta X^\nu \eta_{\kappa\nu} a^{\kappa\delta} X_\delta$$

$$= -T\sqrt{-h}\, h^{\gamma\beta}\partial_\beta X_\kappa a^{\kappa\delta} X_\delta$$

Inserting $h^{\alpha\beta}$: $j^\gamma = -T\left\{-\delta^{\gamma 0}\partial_0 X_\kappa a^{\kappa\delta} X_\delta + \delta^{\gamma 1}\partial_1 X_\kappa a^{\kappa\delta} X_\delta\right\}$

③

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Polyakov action $S_p = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\left(h^{\alpha\beta}\partial_\alpha X^\mu(\sigma)\partial_\beta X^\nu(\sigma)\eta_{\mu\nu}\right)$

a) world sheet reparametrization $\sigma^\alpha \to \sigma'^\alpha(\sigma^\beta)$

$$S_p \longmapsto S_p' = -\frac{T}{2}\int d^2\sigma' \sqrt{-h'}\left(h'^{\alpha\beta}\partial'_\alpha X^\mu(\sigma'(\sigma))\partial'_\beta X^\nu(\sigma'(\sigma))\eta_{\mu\nu}\right)$$

where $h'(\sigma'(\sigma))$, $h(\sigma)$

$$d^2\sigma' = d^2\sigma\left|\det\left(\frac{d\sigma'^\kappa}{d\sigma^\lambda}\right)\right|$$

$$\det h'_{\kappa\lambda} = \det h_{\varsigma\delta}\frac{d\sigma^\varsigma}{d\sigma'^\kappa}\frac{d\sigma^\delta}{d\sigma'^\lambda}$$

$$\partial'_\alpha X^\mu(\sigma'(\sigma)) = \frac{\partial\sigma^\beta}{\partial\sigma'^\alpha}\partial_\beta X^\mu(\sigma)$$

correct trafo for 2nd rank tensor, i.e. $h'_{\alpha\beta} = h_{\gamma\delta}\frac{d\sigma^\gamma}{d\sigma'^\alpha}\frac{d\sigma^\delta}{d\sigma'^\beta}$ or $= h_{\delta\gamma}\frac{d\sigma^\gamma}{d\sigma^\alpha}\frac{d\sigma^\delta}{d\sigma^\beta}$

$\leadsto h_{\alpha\beta} \to \frac{\partial\sigma'^\tau}{\partial\sigma^\alpha}\frac{\partial\sigma'^\varsigma}{\partial\sigma^\beta}h'_{\tau\varsigma}$ multiply h' w/o σ' on rhs.

$$= -\frac{T}{2}\int d^2\sigma \left|\det\frac{d\sigma'}{d\sigma}\right|\sqrt{\det\frac{d\sigma}{d\sigma'}\frac{d\sigma}{d\sigma'}}\sqrt{-h}$$

$\det(AB) = \det A \det B$

$\det A^{-1} = (\det A)^{-1}$

$$\times\, h^{\kappa\lambda}\frac{d\sigma'^\alpha}{d\sigma^\kappa}\frac{d\sigma'^\beta}{d\sigma^\lambda}\partial_\varsigma X^\mu(\sigma)\partial_\delta X^\nu(\sigma)\eta_{\mu\nu}\frac{\partial\sigma^\varsigma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta}$$

$\frac{d\sigma'^\alpha}{d\sigma^\kappa}\frac{d\sigma^\varsigma}{d\sigma'^\alpha} = \delta^\varsigma_\kappa$? $\leadsto$ yes, for the 2x α's the same, then you multiply Jacobian w/ inverse Jacobian

$$\stackrel{\downarrow}{=} -\frac{T}{2}\int d^2\sigma \sqrt{-h}\left(h^{\kappa\lambda}\partial_\kappa X^\mu(\sigma)\partial_\lambda X^\nu(\sigma)\eta_{\mu\nu}\right)$$

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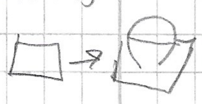
$= S_p$

what is this physically? $\to$ rescaling of distances

b) Weyl transformation $h_{\alpha\beta} \to e^{\phi(\sigma^\alpha)}h_{\alpha\beta}$

$\leadsto h^{\alpha\beta} \to e^{-\phi(\sigma^\alpha)}h^{\alpha\beta}$ s.t. $h^{\alpha\beta}h_{\beta\kappa} = \delta^\alpha_\kappa$



$$\leadsto S_p \longmapsto S_p' = -\frac{T}{2}\int d^2\sigma\sqrt{-e^{2\phi(\sigma^\alpha)}h}\,\Big\{e^{-\phi(\sigma^\alpha)}h^{\alpha\beta}$$

$$\times\,\partial_\alpha X^\mu(\sigma)\partial_\beta X^\nu(\sigma)\eta_{\mu\nu}\Big\}$$

2

$\det(C h_{\alpha\beta}) = C^2\det h_{\alpha\beta}$, as h is a tensor of dim 2

2

$= S_p$

dimension 2? rank is also 2, but $\alpha=0,1$, $\beta=0,1$ does not refer to rank? E.g. $\eta_{\mu\nu}$ Mink. metric rank 2 and dimension 4?

c) If $h_{\alpha\beta} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ not only locally but globally with a parametrization, then the metric would be flat, $\Gamma = 0$ for the Christoffel symbols and $R_{\mu\nu} = 0$ for the Ricci tensor. this is not always true.

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$2g-2 = \int d^2\sigma\sqrt{-h}R^{(2)}$

$\leadsto$ if $R=0$ $\leadsto$ fixed topology, but ...

Why is it not always true?

~~How to fix locally?~~

2) The Riemann tensor in 2 dimensions has the non-vanishing components $C \equiv R_{0101} = -R_{0110} = -R_{1001} = R_{1010}$

Then, we find with $R_{\mu\nu} = R^{\kappa}{}_{\mu\kappa\nu} = g^{\kappa\lambda} R_{\lambda\mu\kappa\nu}$ that

$$R_{00} = g^{\kappa\lambda} R_{\lambda 0 \kappa 0} = g^{11} R_{1010}, \quad R_{11} = g^{\kappa\lambda} R_{\lambda 1 \kappa 1} = g^{00} R_{0101}$$

$$R_{01} = g^{\kappa\lambda} R_{\lambda 0 \kappa 1} = g^{01} R_{1001} = R_{10}$$

and with $R = g^{\beta\nu} R_{\beta\nu}$ that

$$R = g^{00} R_{00} + g^{11} R_{11} + g^{01} R_{01} + g^{10} R_{10}$$

$$= 2 g^{00} g^{11} C - 2 g^{01} g^{10} C = 2C \det g^{-1}$$

with $g^{-1} = g^{\alpha\beta} = \begin{pmatrix} g^{00} & g^{01} \\ g^{10} & g^{11} \end{pmatrix}$, $g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix}$

$$= \frac{1}{\det g} \begin{pmatrix} g_{11} & -g_{01} \\ -g_{10} & g_{00} \end{pmatrix} \rightsquigarrow g g^{-1} = \frac{1}{\det g} \begin{pmatrix} g_{00} g_{11} - g_{10}^2 & 0 \\ 0 & g_{01} g_{10} + g_{00} g_{11} \end{pmatrix} = \mathbb{1}$$

Using Einstein's eq. $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$, we find that

$$T_{00} \propto R_{00} - C (\det g)^{-1} g_{00} = g^{11} C - C g^{11} = 0$$

$$T_{01} \propto R_{01} - C (\det g)^{-1} g_{01} = -g^{01} C + C g^{01} = 0$$

$$= T_{10}$$

$$T_{11} \propto R_{11} - C (\det g)^{-1} g_{11} = g^{00} C - C g^{00} = 0$$

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For 2D: $R_{\mu\nu\rho\sigma} = \frac{R}{2} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma})$

$$R_{\mu\nu} = \frac{R}{2} g_{\mu\nu}$$

$$R_{\mu\nu} - \frac{g_{\mu\nu} R}{2} = 0 \rightarrow T_{\mu\nu} = 0$$

3) $\tau \in \mathbb{R}$, $\sigma \in [0,\pi]$, $\tau \in (-\infty, \infty)$

Why these boundaries?
⇝ string defined like this on WS; τ: time and additionally σ as curve par.

w/ Polyakov or w/ Nambu Goto?
⇝ are equivalent but Polyakov easier

a) gauge-fixed metric: $h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ⇝ $\det h = -1$

$$S_p = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$$= -\frac{T}{2}\int d^2\sigma \{h^{00}\partial_0 X^\mu \partial_0 X_\mu + h^{11}\partial_1 X^\mu \partial_1 X_\mu\}$$

(1)

$$\cancel{=} -\frac{T}{2}\int d^2\sigma \{X'^2 - \dot{X}^2\} = \frac{T}{2}\int d^2\sigma \{\dot{X}^2 - X'^2\}$$

Also possible to do b) w/o gauge fixed metric? ⇝ gives other result e.o.m. $\square X^\mu$ w/ $\square X^\mu = \frac{1}{\sqrt{-h}}\partial_\alpha(\sqrt{-h}\, h^{\alpha\beta}\partial_\beta X^\mu)$

b) We will use the principle of least action, i.e. $\delta S = 0$ for S_p:

$$0 \overset{!}{=} \delta S = \frac{d}{d\lambda} S[X + \lambda \delta X]\Big|_{\lambda=0}$$

Way w/o λ? ⇝ $\delta S_p = -\frac{1}{2\pi\alpha'}\int d^2\sigma \Big[\frac{\partial(\dot{X}^2 - X'^2)}{\partial \dot{X}^\mu}\delta(\dot{X}^\mu) + \frac{\partial(\dot{X}^2 - X'^2)}{\partial X'^\mu}\delta X'^\mu + \cancel{}\,\delta X^\mu\Big]$

$$= \frac{T}{2}\frac{d}{d\lambda}\int d^2\sigma \Big\{\frac{d}{d\tau}(X_\mu + \lambda\delta X_\mu)\Big\}\Big\{\frac{d}{d\tau}(X^\mu + \lambda \delta X^\mu)\Big\} - \Big\{\frac{d}{d\sigma}(X_\mu + \lambda\delta X_\mu)\Big\}\Big\{\frac{d}{d\sigma}(X^\mu + \lambda\delta X^\mu)\Big\}\Big|_{\lambda=0}$$

$$= \frac{T}{2}\int d^2\sigma \{2(\delta\dot{X}_\mu)\dot{X}^\mu - 2(\delta X_\mu)' X^{\mu\prime}\}$$

$$= T\int d^2\sigma \{\dot{X}^\mu(\delta\dot{X}_\mu) - X^{\mu\prime}(\delta X_\mu)'\}$$

$$= T\int d^2\sigma \Big[\frac{d}{d\tau}\{\dot{X}^\mu(\delta X_\mu)\} - \ddot{X}^\mu(\delta X_\mu) - \frac{d}{d\sigma}\{X^{\mu\prime}(\delta X_\mu)\} + X^{\mu\prime\prime}(\delta X_\mu)\Big]$$

$$= T\Big\{\int d\sigma\, \dot{X}^\mu \delta X_\mu\Big|_{\tau=-\infty}^{\infty} - \int d^2\sigma\, \ddot{X}^\mu(\delta X_\mu) - \int d\tau\, X^{\mu\prime}(\delta X_\mu)\Big|_{\sigma=0}^{\pi} + \int d^2\sigma\, X^{\mu\prime\prime}(\delta X_\mu)\Big\}$$

Why $\int \tau' \ldots$? (eq. (12)) ⇝ should be $d\tau$

(2) ⇝ e.o.m. $X^{\mu\prime\prime} - \ddot{X}^\mu = 0$

~~and~~ a boundary term $\mathcal{B} = -T\int d\tau \{X^{\mu\prime}(\delta X_\mu)|_{\sigma=\pi} - X^{\mu\prime}(\delta X_\mu)|_{\sigma=0}\}$

c) For the boundary term to vanish, we have three possibilities:

(2)

- (b) $X^{\mu\prime}(\sigma,\tau) = 0$ for $\sigma = \pi, 0$ ← open string; ~~closed string~~ trivial $\left(\hat{=} \frac{\partial \mathcal{L}}{\partial X'^\mu} = 0\right)$ Neumann, can move freely

- (c) $X^\mu|_{\sigma=0} = X^\mu_0$, $X^\mu|_{\sigma=\pi} = X^\mu_\pi$ ← open string; const (& τ) ⇝ variation $\delta X_\mu = 0$ at $\sigma = \pi, 0$ ⇝ vanishes ~~Neum.~~ Dirichlet ⇝ ends fixed

- $X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau)$ ← closed string ⇝ $\delta X^\mu(\sigma,\tau) = \delta X^\mu(\sigma+\pi,\tau)$ ← Closed string

  $X^{\mu\prime}(\sigma,\tau) = X^{\mu\prime}(\sigma+\pi,\tau)$ ⇝ exactly cancel

d) general solution to e.o.m. $X^\mu(\sigma,\tau) = f(\sigma)g(\tau)$

Closed strings: $X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau) \Leftrightarrow f(\sigma)g(\tau) = f(\sigma+\pi)g(\tau)$

$$\frac{\partial^2 X^\mu(\sigma,\tau)}{\partial\sigma^2} = \frac{\partial^2 f(\sigma)}{\partial^2\sigma} g(\tau) \quad , \quad \frac{\partial^2 X^\mu(\sigma,\tau)}{\partial\tau^2} = f(\sigma)\frac{\partial^2 g(\tau)}{\partial\tau^2}$$

and $\ddot{X}^\mu(\sigma,\tau) = X^{\mu\,\prime\prime}(\sigma,\tau) \leadsto \frac{\partial^2 f(\sigma)}{\partial^2\sigma} g(\tau) = f(\sigma)\frac{\partial^2 g(\tau)}{\partial\tau^2}$

$$\leadsto \frac{1}{f(\sigma)}\frac{\partial^2 f(\sigma)}{\partial^2\sigma} = \frac{1}{g(\tau)}\frac{\partial^2 g(\tau)}{\partial\tau^2} = \text{const}$$ (both indep. of other variable)

$$\leadsto \frac{\partial^2 f(\sigma)}{\partial^2\sigma} = C f(\sigma) \text{ and } \frac{\partial^2 g(\tau)}{\partial\tau^2} = C g(\tau)$$

One then finds $f(\sigma) = A(\tau)e^{\pm\sqrt{C}\sigma}$, $g(\tau) = B(\sigma)e^{\pm\sqrt{C}\tau}$ two possible

$\leadsto X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau)$ implies

factor i

$$f(\sigma+\pi) = f(\sigma) \leadsto \sqrt{C}\pi = 2im\pi, \; m\in\mathbb{Z}$$

$$\leadsto C = -4m^2$$

$\uparrow (-)$

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why is this the general solution? $\leadsto$ Ansatz of sep. of vars; turns out to work here.

why $C=-4m^2$ Equivalent to $4m^2$ for $m\in\mathbb{Z}$... $\leadsto$ forgot i

Also different Ansatz possible for $f(\sigma), g(\tau)$ w/o exp?

$$\leadsto f = \sum_{m\neq 0} A_m e^{2im\sigma} + D_0 = \sum_{m>0} A_{-m}e^{-2im\sigma} + A_m e^{2im\sigma} + D_0$$

$$g = \sum_{m\neq 0} B_m e^{2im\tau} + C_1\tau + C_0 = \sum_{m>0} B_{-m}e^{-2im\tau} + B_m e^{2im\tau} + C_1\tau + C_0$$

$X^\mu = f\cdot g =$

?

e) We now change to light cone coordinates,

$$\sigma^{\pm} = \tau \pm \sigma \quad \leadsto \quad \tau = \frac{1}{2}(\sigma^+ + \sigma^-)$$
$$\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$$

$$\frac{\partial}{\partial\sigma^+} = \frac{\partial}{\partial\tau}\frac{\partial\tau}{\partial\sigma^+} + \frac{\partial}{\partial\sigma}\frac{\partial\sigma}{\partial\sigma^+} = \frac{1}{2}\partial_\tau + \frac{1}{2}\partial_\sigma \quad \Bigg| \quad \partial_+ = \frac{1}{2}(\partial_\tau + \partial_\sigma), \quad \partial_- = \frac{1}{2}(\partial_\tau - \partial_\sigma) \quad \Bigg\} \quad \partial_\tau = \partial_+ + \partial_-, \quad \partial_\sigma = \partial_+ - \partial_-$$

✓ We find $\partial_+\partial_- X^\mu(\sigma^+,\sigma^-) = \frac{1}{4}(\partial_\tau^2 - \partial_\sigma^2) X^\mu(\sigma,\tau) = 0$

$\partial_+\partial_- X^\mu = 0$ only reason for $X^\mu = X_R^\mu + X_L^\mu$? $\leadsto$ yes

and thus

$$X^\mu(\sigma^+,\sigma^-) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$$

for arbitrary fields $X_R^\mu(\tau-\sigma)$, $X_L^\mu(\tau+\sigma)$, the right- and left-movers. ✓

X_R right-mover and X_L left-mover? $\leadsto$ yes

As we are looking for general closed string solutions, we demand $X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau)$

For a periodic function the derivative is also periodic, as

$$f'(x+k) = \lim_{h\to 0}\frac{f(x+k+h) - f(x+k)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$$

such that $\partial_\sigma X^\mu(\sigma+\pi,\tau) = \partial_\sigma X^\mu(\sigma,\tau)$

$$\Leftrightarrow (\partial_+ - \partial_-)\{X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)\}$$
$$= (\partial_+ - \partial_-)\{X_R^\mu(\tau-\sigma-\pi) + X_L^\mu(\tau+\sigma+\pi)\}$$
$$\Leftrightarrow \partial_+ X_L^\mu(\sigma^+) - \partial_- X_R^\mu(\sigma^-)$$
$$= \partial_+ X_L^\mu(\sigma^+ + \pi) - \partial_- X_R^\mu(\sigma^- - \pi)$$

Argument really correct? Or even necessary? ?

As σ^+ and σ^- independent $\leadsto$ $\partial_+ X_L^\mu(\sigma^+)$ and $\partial_- X_R^\mu(\sigma^-)$ both π-periodic seperately.

How did the string length get here? ? $l_s^2 = 4\pi^2\alpha'$? $T = 1/2\pi\alpha'$?

$$\leadsto \partial_+ X_L^\mu(\sigma^+) = l_s \sum_{n=-\infty}^{\infty} \bar\alpha_n^\mu e^{-2in\sigma^+} = l_s\Big\{\bar\alpha_0^\mu + \sum_{n\neq 0}\bar\alpha_n^\mu e^{-2in\sigma^+}\Big\}$$
$$\partial_- X_R^\mu(\sigma^-) = l_s \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-2in\sigma^-} = l_s\Big\{\alpha_0^\mu + \sum_{n\neq 0}\alpha_n^\mu e^{-2in\sigma^-}\Big\}$$

✓ $\alpha_0^\mu = \bar\alpha_0^\mu$ has to hold? $\leadsto$ yes.

$$\leadsto X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu\sigma^+ + \frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}\bar\alpha_n^\mu e^{-2in\sigma^+}$$
$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu\sigma^- + \frac{i}{2}l_s\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}$$

with $p^\mu = \frac{2}{l_s}\alpha_0^\mu = \frac{2}{l_s}\bar\alpha_0^\mu$

$$\Rightarrow X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$$

$$= \underbrace{x^\mu + l_s^2 p^\mu}_{\text{CM motion}} + \underbrace{\frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n} \left\{ \alpha_n^\mu e^{2in\sigma} + \bar{\alpha}_n^\mu e^{-2in\sigma} \right\} e^{-2in\tau}}_{\text{oscillation of the string}}$$

For $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ and $\bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^*$, we find

$$\sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau-\sigma)} \supset \frac{1}{n} \left\{ \alpha_n^\mu e^{-2in(\tau-\sigma)} \underset{\substack{\uparrow \\ \frac{1}{n} \text{ changes sign}}}{-} (\alpha_n^\mu)^* e^{+2in(\tau-\sigma)} \right\} \in \mathbb{C}\setminus\mathbb{R}$$

$$\sum_{n\neq 0} \frac{1}{n} \bar{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \supset \frac{1}{n} \left\{ \bar{\alpha}_n^\mu e^{-2in(\tau+\sigma)} - (\bar{\alpha}_n^\mu)^* e^{2in(\tau+\sigma)} \right\} \in \mathbb{C}\setminus\mathbb{R}$$

Such that with the factor of (i) in front of the sum, it becomes real. As $x^\mu \in \mathbb{R}$ and $p^\mu \in \mathbb{R}$, we have $X^\mu(\sigma,\tau) \in \mathbb{R}$

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