String theory Exercise 2 Homework Marvin Zanke

10 + 5 + 9 = 24

23.10.2018

1) Having $X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$

with
$$X_R^\mu = \tfrac{1}{2}x^\mu + \tfrac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \tfrac{i}{2}l_s \sum_{n\neq 0}\tfrac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}$$
$$X_L^\mu = \tfrac{1}{2}x^\mu + \tfrac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \tfrac{i}{2}l_s \sum_{n\neq 0}\tfrac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}$$

we find

$$X^\mu(\sigma,\tau) = x^\mu + l_s^2 p^\mu \tau + \tfrac{i}{2}l_s \sum_{n\neq 0}\tfrac{1}{n}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right) \times e^{-2in\tau}$$

and

$$P^\mu(\sigma,\tau) = T\dot{X}^\mu(\sigma,\tau) = Tl_s^2 p^\mu + Tl_s \sum_{n\neq 0}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right)e^{-2in\tau}$$

Everything here only valid for this specific gauge fixed action?

we have to gauge fix to solve it

$$\Rightarrow \boxed{\frac{1}{\pi}\int_0^\pi X^\mu(\sigma,0)\,d\sigma = x^\mu}$$ as

$$\int_0^\pi d\sigma\, e^{\pm 2in\sigma} = \pm\frac{1}{2in}\left[e^{\pm 2in\sigma}\right]\Big|_0^\pi = 0 \quad (*)$$

and $\boxed{\frac{1}{\pi}\int_0^\pi P^\mu(\sigma,0)\,d\sigma = Tl_s^2 p^\mu}$ from the same argument

Also $\boxed{\frac{2}{i\pi l_s}\int_0^\pi d\sigma X^\mu(\sigma,0)e^{-2ik\sigma} = \frac{1}{k}\left(\alpha_k^\mu - \tilde{\alpha}_{-k}^\mu\right)}$

because $\int_0^\pi x^\mu e^{-2ik\sigma}d\sigma = 0$ (see (*))

and $\int_0^\pi d\sigma\left\{\tfrac{i}{2}l_s \sum_{n\neq 0}\tfrac{1}{n}\left(\alpha_n^\mu e^{2i\sigma(n-k)} + \tilde{\alpha}_n^\mu e^{-2i\sigma(n+k)}\right)\right\}$

$\overset{n\pm k\neq 0 \Rightarrow \text{see } (*)}{=} \tfrac{i}{2}l_s\tfrac{1}{k}\left(\alpha_k^\mu - \tilde{\alpha}_{-k}^\mu\right)\pi$

and $\boxed{\frac{1}{Tl_s\pi}\int_0^\pi d\sigma\, P^\mu(\sigma,0)e^{-2ik\sigma} = \alpha_k^\mu + \tilde{\alpha}_{-k}^\mu}$

because $Tl_s^2\int_0^\pi p^\mu e^{-2ik\sigma} = 0$. (see (*))

and $\int_0^\pi d\sigma\left\{Tl_s\sum_{n\neq 0}\left(\alpha_n^\mu e^{2i\sigma(n-k)} + \tilde{\alpha}_n^\mu e^{-2i\sigma(n+k)}\right)\right\}$

$= Tl_s\left(\alpha_k^\mu + \tilde{\alpha}_{-k}^\mu\right)\pi$

Using these relations, we find:

$$\alpha_k^\mu = \frac{1}{2}\left\{\frac{2k}{i\pi l_s}\int_0^\pi d\sigma\, X^\mu(\sigma,0)\, e^{-2ik\sigma} + \frac{1}{T l_s \pi}\int d\sigma\, P^\mu(\sigma,0)\, e^{-2ik\sigma}\right\}$$

$$\tilde{\alpha}_{-k}^\mu = \frac{1}{2}\left\{\frac{1}{T l_s \pi}\int_0^\pi d\sigma\, P^\mu(\sigma,0)\, e^{-2ik\sigma} - \frac{2k}{i\pi l_s}\int_0^\pi d\sigma\, X^\mu(\sigma,0)\, e^{-2ik\sigma}\right\}$$

and then:

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \frac{1}{4}\left\{\frac{2m}{i\pi l_s}\int_0^\pi d\sigma\, X^\mu(\sigma,0)\, e^{-2im\sigma} + \frac{1}{T l_s \pi}\int_0^\pi d\sigma\, P^\mu(\sigma,0)\, e^{-2im\sigma},\right.$$

$$\left.\frac{2n}{i\pi l_s}\int_0^\pi d\sigma'\, X^\nu(\sigma',0)\, e^{-2in\sigma'} + \frac{1}{T l_s \pi}\int_0^\pi d\sigma'\, P^\nu(\sigma',0)\, e^{-2in\sigma'}\right\} \quad (\ast\ast)$$

$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = 0 = \{P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\}$$

Only possible at same τ? No bracket w/ τ and τ' as for σ, σ' ⇝ Hamilton formalism; take for fixed time and time evolution given by Hamiltonian (cf. canonical mech.)

$$= \frac{1}{4}\left[\frac{2m}{i\pi^2 l_s^2 T}\int_0^\pi d\sigma\int_0^\pi d\sigma'\, \{X^\mu(\sigma,0), P^\nu(\sigma',0)\}\, e^{-2im\sigma}\, e^{-2in\sigma'}\right.$$

$$\left.+ \frac{2n}{i\pi^2 l_s T}\int_0^\pi d\sigma\int_0^\pi d\sigma'\, \{P^\mu(\sigma,0), X^\nu(\sigma',0)\}\, e^{-2im\sigma}\, e^{-2in\sigma'}\right]$$

$$-\{P^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \eta^{\mu\nu}\,\delta(\sigma-\sigma') = +\{X^\nu(\sigma',\tau), P^\mu(\sigma,\tau)\}$$

wrong way around on the sheet? ⇝ yes!

$$= \frac{m}{2i\pi^2 l_s^2 T}\int_0^\pi d\sigma\, (+\eta^{\mu\nu})\, e^{-2i\sigma(m+n)} + \frac{n}{2i\pi^2 l_s^2 T}\int_0^\pi d\sigma\, (+\eta^{\mu\nu})\, e^{-2i\sigma(m+n)}$$

vanishes if $m+n\neq 0$

$$= \frac{2m}{2i\pi^2 l_s^2 T}\,\eta^{\mu\nu}\,\pi\,\delta_{m+n,0} = \frac{-im}{\pi l_s^2 T}\,\eta^{\mu\nu}\,\delta_{m+n,0}$$

$$\boxed{T = \frac{1}{\pi l_s^2}}$$

$$\stackrel{\downarrow}{=} -im\,\eta^{\mu\nu}\,\delta_{m+n,0}$$

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Is $T = \frac{1}{\pi l_s^2}$ the correct def.? In the lecture: $l_s = 2\pi\sqrt{\alpha'}$, $\alpha' = \frac{1}{4\pi T}$ ⇝ $T = \pi/l_s^2$ ⇒ wrong result?!

Analogously, we find:

$$\{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\} = \frac{1}{4}\Big\{ \frac{1}{T l_s \pi}\int_0^\pi d\sigma\, P^\mu(\sigma,0)\, e^{+2im\sigma} + \frac{2m}{i\pi l_s}\int_0^\pi d\sigma\, X^\mu(\sigma,0)\, e^{+2im\sigma},$$

$$\frac{1}{T l_s \pi}\int_0^\pi d\sigma'\, P^\nu(\sigma',0)\, e^{+2in\sigma'} + \frac{2n}{i\pi l_s}\int_0^\pi d\sigma'\, X^\nu(\sigma',0)\, e^{+2in\sigma'}\Big\}$$

exactly same integral as (**) $w/ \sigma^{(1)} \mapsto -\sigma^{(1)}$

$$= -im\,\eta^{\mu\nu}\,\delta_{m+n,0}$$

Valid argument $w/ \sigma \mapsto -\sigma$, as then $\{P^\mu(-\sigma,0), X^\nu(-\sigma',0)\}$ etc.?

$\rightsquigarrow$ okay

$$\{\alpha^\mu_m, \tilde{\alpha}^\nu_n\} = \frac{1}{4}\Big\{ \frac{2m}{i\pi l_s}\int_0^\pi d\sigma\, X^\mu(\sigma,0)\, e^{-2im\sigma} + \frac{1}{T l_s \pi}\int_0^\pi d\sigma\, P^\mu(\sigma,0)\, e^{-2im\sigma},$$

$$\frac{2n}{i\pi l_s}\int_0^\pi d\sigma'\, X^\nu(\sigma',0)\, e^{+2in\sigma'} + \frac{1}{T l_s \pi}\int_0^\pi d\sigma'\, P^\nu(\sigma',0)\, e^{+2in\sigma'}\Big\}$$

$$= \frac{1}{4}\Big[\frac{2m}{i\pi^2 l_s^2 T}\int_0^\pi d\sigma \int_0^\pi d\sigma'\, \{X^\mu(\sigma,0), P^\nu(\sigma',0)\}\, e^{-2im\sigma}\, e^{+2in\sigma'}$$

$$+ \frac{2n}{i\pi^2 l_s^2 T}\int_0^\pi d\sigma \int_0^\pi d\sigma'\, \{P^\mu(\sigma,0), X^\nu(\sigma',0)\}\, e^{-2im\sigma}\, e^{+2in\sigma'}\Big]$$

$$= \frac{m}{2i\pi^2 l_s^2 T}\int_0^\pi d\sigma\, (+\eta^{\mu\nu})\, e^{-2i\sigma(m-n)} + \frac{n}{2i\pi^2 l_s^2 T}\int_0^\pi d\sigma\, (\eta^{\mu\nu})\, e^{-2i\sigma(m-n)}$$

vanishes if $m \neq n$

$$= \frac{1}{2i\pi^2 l_s^2 T}\,\eta^{\mu\nu}\,\{\pi m - \pi n\}\,\delta_{m,n} = 0$$

$$\{x^\mu, x^\nu\} = \frac{1}{\pi^2}\Big\{\int_0^\pi X^\mu(\sigma,0)\, d\sigma, \int_0^\pi X^\nu(\sigma',0)\, d\sigma'\Big\}$$

$$= \frac{1}{\pi^2}\int_0^\pi d\sigma \int_0^\pi d\sigma'\, \{X^\mu(\sigma,0), X^\nu(\sigma',0)\} = 0$$

$$\{p^\mu, p^\nu\} = \frac{1}{\pi^2 T^2 l_s^4}\Big\{\int_0^\pi d\sigma\, P^\mu(\sigma,0), \int_0^\pi d\sigma'\, P^\nu(\sigma',0)\Big\}$$

$$= \frac{1}{\pi^2 T^2 l_s^4}\int_0^\pi d\sigma \int_0^\pi d\sigma'\, \{P^\mu(\sigma,0), P^\nu(\sigma',0)\} = 0$$

$$\{x^\mu, p^\nu\} = \frac{1}{\pi^2 T l_s^2}\Big\{\int_0^\pi d\sigma\, X^\mu(\sigma,0), \int_0^\pi d\sigma'\, P^\nu(\sigma',0)\Big\}$$

$$= \frac{1}{\pi^2 T l_s^2}\int_0^\pi d\sigma \int_0^\pi d\sigma'\, \{X^\mu(\sigma,0), P^\nu(\sigma',0)\}$$

$$= \frac{1}{\pi^2 T l_s^2}\int_0^\pi d\sigma\, \eta^{\mu\nu} \overset{T = \frac{1}{\pi l_s^2}}{=} \eta^{\mu\nu}$$

$$\{X^\mu, \tilde{\alpha}_n^\nu\} = \frac{1}{2\pi}\left\{\int_0^\pi d\sigma\, X^\mu(\sigma,0),\ \frac{1}{T\ell_s\pi}\int_0^\pi d\sigma'\, P^\nu(\sigma',0)e^{+2in\sigma'} + \frac{2n}{i\pi\ell_s}\int_0^\pi d\sigma'\, X^\nu(\sigma',0)e^{+2in\sigma'}\right\}$$

$$= \frac{1}{2\pi^2 T\ell_s}\int_0^\pi d\sigma\int_0^\pi d\sigma'\,\{X^\mu(\sigma,0), P^\nu(\sigma',0)\}e^{+2in\sigma'}$$

$$\int_0^\pi d\sigma\, e^{2in\sigma} = 0. \text{ since } n \text{ is integer}$$

$$= \frac{1}{2\pi^2 T\ell_s}\int_0^\pi d\sigma\,\eta^{\mu\nu}e^{+2in\sigma} = \frac{1}{2\pi T\ell_s}\eta^{\mu\nu}\delta_{n,0} = \cancel{\frac{1}{2}\eta^{\mu\nu}\delta_{n,0}}$$

For n=0 does not vanish? ~~w~~ no $e^{-2in\sigma}$ in the sum, it's again $d\sigma = p$

$$\{X^\mu, \alpha_n^\nu\} = \frac{1}{2\pi}\left\{\int_0^\pi d\sigma\, X^\mu(\sigma,0),\ \frac{2n}{i\pi\ell_s}\int_0^\pi d\sigma'\, X^\nu(\sigma',0)e^{-2in\sigma'} + \frac{1}{T\ell_s\pi}\int d\sigma'\, P^\nu(\sigma',0)e^{-2in\sigma'}\right\}$$

and $e^{2in\pi} = 1$

$$= \frac{1}{2\pi^2 T\ell_s}\int_0^\pi d\sigma\int_0^\pi d\sigma'\,\{X^\mu(\sigma,0), P^\nu(\sigma',0)\}e^{-2in\sigma'}$$

$$= \frac{1}{2\pi T\ell_s}\eta^{\mu\nu}\delta_{n,0} = \frac{1}{2}\eta^{\mu\nu}\delta_{n,0}$$

$$\{P^\mu, \tilde{\alpha}_n^\nu\} = \frac{1}{2\pi T\ell_s^2}\left\{\int_0^\pi d\sigma\, P^\mu(\sigma,0),\ \frac{1}{T\ell_s\pi}\int_0^\pi d\sigma'\, P^\nu(\sigma',0)e^{+2in\sigma'} + \frac{2n}{i\pi\ell_s}\int_0^\pi d\sigma'\, X^\nu(\sigma',0)e^{+2in\sigma'}\right\}$$

$$= \frac{n}{i\pi^2\ell_s^3 T}\int_0^\pi d\sigma\int_0^\pi d\sigma'\,\{P^\mu(\sigma,0), X^\nu(\sigma',0)\}\cancel{e^{2in\sigma'}}$$

same as above ?

$$= \frac{-n}{i\pi\ell_s^3 T}\eta^{\mu\nu}\delta_{n,0} = 0$$

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$$\{P^\mu, \alpha_n^\nu\} = \frac{1}{2\pi T\ell_s^2}\left\{\int_0^\pi d\sigma\, P^\mu(\sigma,0),\ \frac{2n}{i\pi\ell_s}\int_0^\pi d\sigma'\, X^\nu(\sigma',0)e^{-2in\sigma'} + \frac{1}{T\ell_s\pi}\int_0^\pi d\sigma'\, P^\nu(\sigma',0)e^{-2in\sigma'}\right\}$$

$$= \frac{n}{i\pi^2 T\ell_s^3}\int_0^\pi d\sigma\int_0^\pi d\sigma'\,\{P^\mu(\sigma,0), X^\nu(\sigma',0)\}\cancel{e^{-2in\sigma'}}$$

$$= \frac{n}{i\pi T\ell_s^3}\eta^{\mu\nu}\delta_{n,0} = 0$$

What for was $\delta(\sigma-\sigma') = \frac{1}{\pi}\sum_n e^{2in(\sigma-\sigma')}$ given? ~~w~~ if calculated by inserting X^μ and P^ν (like tried before)

1.2) current to charge

$$J^{\mu\nu}(\sigma,\tau) = X^\mu P^\nu - X^\nu P^\mu$$

$\mathcal{J}^{\mu\nu}$ or $J^{\mu\nu}$ the ang. mom.? → the integrated $\mathcal{J}$, in order to be conserved charge, need the integral

$$X^\mu(\sigma,\tau) = x^\mu + l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right) e^{-2in\tau}$$

$$P^\mu(\sigma,\tau) = T\dot{X}^\mu(\sigma,\tau) = Tl_s^2 p^\mu + Tl_s \sum_{n\neq 0}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right) e^{-2in\tau}$$

$$\rightsquigarrow X^\mu P^\nu = Tl_s^2 x^\mu p^\nu + Tl_s x^\mu \sum_{n\neq 0}\left(\alpha_n^\nu e^{2in\sigma} + \tilde{\alpha}_n^\nu e^{-2in\sigma}\right)e^{-2in\tau}$$
$$+ Tl_s^4 p^\mu p^\nu \tau + Tl_s^3 p^\mu \tau \sum_{n\neq 0}\left(\alpha_n^\nu e^{2in\sigma} + \tilde{\alpha}_n^\nu e^{-2in\sigma}\right)e^{-2in\tau}$$
$$+ \frac{i}{2} Tl_s^3 \sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right)e^{-2in\tau}\, p^\nu$$
$$+ \frac{i}{2} Tl_s^2 \sum_{n,m\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{2in\sigma} + \tilde{\alpha}_n^\mu e^{-2in\sigma}\right)\left(\alpha_m^\nu e^{2im\sigma} + \tilde{\alpha}_m^\nu e^{-2im\sigma}\right)e^{-2i\tau(n+m)}$$

$$\rightsquigarrow \mathcal{J}^{\mu\nu} = \int_0^\pi J^{\mu\nu}(\sigma,\tau)\,d\sigma = \int_0^\pi d\sigma\,(X^\mu P^\nu - X^\nu P^\mu) = \underbrace{\int_0^\pi d\sigma\, X^\mu P^\nu}_{(*)} - (\mu \leftrightarrow \nu)$$

$$(*) = T\pi l_s^2 x^\mu p^\nu + T\pi l_s^4 p^\mu p^\nu \tau$$
$$+ \frac{i}{2} Tl_s^2 \int_0^\pi d\sigma \sum_{n,m\neq 0}\frac{1}{n}\Big\{\alpha_n^\mu \alpha_m^\nu e^{2i\sigma(n+m)} + \alpha_n^\mu \tilde{\alpha}_m^\nu e^{2i\sigma(n-m)}$$
$$+ \tilde{\alpha}_n^\mu \alpha_m^\nu e^{2i\sigma(m-n)} + \tilde{\alpha}_n^\mu \tilde{\alpha}_m^\nu e^{-2i\sigma(m+n)}\Big\} e^{-2i\tau(n+m)}$$

$$= x^\mu p^\nu + l_s^2 p^\mu p^\nu \tau + \frac{i\pi}{2\pi}\sum_{n\neq 0}\frac{1}{n}\Big\{\alpha_n^\mu \alpha_{-n}^\nu + \alpha_n^\mu \tilde{\alpha}_n^\nu e^{-4i\tau n}$$
$$+ \tilde{\alpha}_n^\mu \alpha_n^\nu e^{-4i\tau n} + \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu\Big\}$$

$$= x^\mu p^\nu + l_s^2 p^\mu p^\nu \tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\Big\{\alpha_n^\mu \alpha_{-n}^\nu + \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu$$
$$+ e^{-4i\tau n}\left(\tilde{\alpha}_n^\mu \alpha_n^\nu + \alpha_n^\mu \tilde{\alpha}_n^\nu\right)\Big\}$$

$$\rightsquigarrow \mathcal{J}^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\Big\{\alpha_n^\mu \alpha_{-n}^\nu - \alpha_n^\nu \alpha_{-n}^\mu$$
$$+ \tilde{\alpha}_n^\mu \tilde{\alpha}_{-n}^\nu - \tilde{\alpha}_n^\nu \tilde{\alpha}_{-n}^\mu\Big\}$$

$\frac{1}{n} < 0$ for $n<0$

e.g. $\alpha_{-n}^\mu \alpha_n^\nu$ also in $\alpha_n^\mu \alpha_{-n}^\nu$ for negative n ⇝ x 2

$$= x^\mu p^\nu - x^\nu p^\mu - i\sum_{n>0}\frac{1}{n}\Big\{\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu\Big\}$$
$$- i\sum_{n>0}\frac{1}{n}\Big\{\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu\Big\}$$

$$=: l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu}$$

Also, $P^\mu = \int_0^\pi d\sigma\, P^\mu(\tau,\sigma)$ and $J^{\mu\nu}$ generate the Poincaré algebra:

$$\{P^\mu, P^\nu\} = \int_0^\pi d\sigma \int_0^\pi d\sigma' \{P^\mu(\tau,\sigma), P^\nu(\tau,\cancel{\sigma'})\} = 0$$

✓ "Generate" the Poinc. algebra? → current from the algebra → cons. charge generates the alg.

$$\{P^\mu, J^{\rho\sigma}\} = \int_0^\pi d\sigma \int_0^\pi d\sigma' \{P^\mu(\tau,\sigma), X^\rho(\tau,\sigma')P^\sigma(\tau,\sigma') - X^\sigma(\tau,\sigma')P^\rho(\tau,\sigma')\}$$

$\{A,BC\} = B\{A,C\} + \{A,B\}C$

$$= \int_0^\pi d\sigma \int_0^\pi d\sigma' \Big[X^\rho(\tau,\sigma')\{P^\mu(\tau,\sigma), P^\sigma(\tau,\sigma')\} + \{P^\mu(\tau,\sigma), X^\rho(\tau,\sigma')\}P^\sigma(\tau,\sigma')$$
$$- X^\sigma(\tau,\sigma')\{P^\mu(\tau,\sigma), P^\rho(\tau,\sigma')\} - \{P^\mu(\tau,\sigma), X^\sigma(\tau,\sigma')\}P^\rho(\tau,\sigma')\Big]$$

$$= \int_0^\pi d\sigma \{(-\eta^{\mu\rho})P^\sigma(\tau,\sigma) + \eta^{\mu\sigma}P^\rho(\tau,\sigma)\}$$

$$= \eta^{\mu\sigma}P^\rho - \eta^{\mu\rho}\cancel{P^\sigma}$$

$$\{J^{\mu\nu}, J^{\rho\sigma}\} = \int_0^\pi d\sigma \int_0^\pi d\sigma' \{X^\mu P^\nu - X^\nu P^\mu, X^\rho P^\sigma - X^\sigma P^\rho\}$$

where on the right side of the P.B. the argument is σ',
while on the left side it is σ.

$$= \int_0^\pi d\sigma \int_0^\pi d\sigma' \Big[\{X^\mu P^\nu, X^\rho P^\sigma\} - \{X^\mu P^\nu, X^\sigma P^\rho\}$$
$$- \{X^\nu P^\mu, X^\rho P^\sigma\} + \{X^\nu P^\mu, X^\sigma P^\rho\}\Big]$$

$$\int_0^\pi d\sigma \int_0^\pi d\sigma' \{X^\mu P^\nu, X^\rho P^\sigma\} = \int_0^\pi d\sigma \int_0^\pi d\sigma' \Big[X^\mu\{P^\nu, X^\rho P^\sigma\} + \{X^\mu, X^\rho P^\sigma\}P^\nu\Big]$$

$$= \int_0^\pi d\sigma \int_0^\pi d\sigma' \Big[X^\mu\{P^\nu, X^\rho\}P^\sigma + X^\rho\{X^\mu, P^\sigma\}P^\nu\Big]$$

$$= \int_0^\pi d\sigma \Big[X^\mu(-\eta^{\nu\rho})P^\sigma + X^\rho\eta^{\mu\sigma}P^\nu\Big]$$

$$= \int_0^\pi d\sigma \Big[\eta^{\mu\sigma}X^\rho P^\nu - \eta^{\nu\rho}X^\mu P^\sigma - \eta^{\mu\rho}X^\sigma P^\nu + \eta^{\nu\sigma}X^\mu P^\rho$$
$$- \eta^{\nu\sigma}X^\rho P^\mu + \eta^{\mu\rho}X^\nu P^\sigma + \eta^{\nu\rho}X^\sigma P^\mu - \eta^{\mu\sigma}X^\nu P^\rho\Big]$$

$$= \eta^{\mu\rho}J^{\nu\sigma} + \cancel{\eta^{\nu\sigma}J^{\mu\rho}} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\sigma}J^{\nu\rho}$$

2) Open string rotating at a constant velocity in (x^1, x^2)-plane. x^0 is the time coordinate.

$$X^0 = A\tau, \quad X^1 = A\cos\tau\cos\sigma, \quad X^2 = A\sin\tau\cos\sigma, \quad X^i = 0, \; i = 3, \dots, D$$

Gauge fixed WS metric only? ⇝ also true for induced metric.

a) Eq. of motion: $0 = \frac{1}{\sqrt{h}}\partial_\alpha\left(\sqrt{h}\, h^{\alpha\beta}\partial_\beta X^\mu\right) = 0$

for gauge fixed WS metric $h^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, this reduces to

also yields this for pullback WS metric (see Mathematica file)

$$\ddot{X}^\mu - X''^\mu = 0$$

$$\dot{X}^\mu = \begin{pmatrix} A \\ -A\sin\tau\cos\sigma \\ A\cos\tau\cos\sigma \\ 0 \\ \vdots \end{pmatrix}, \quad X'^\mu = \begin{pmatrix} 0 \\ -A\cos\tau\sin\sigma \\ -A\sin\tau\sin\sigma \\ 0 \\ \vdots \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\leadsto \quad \ddot{X}^\mu = \begin{pmatrix} 0 \\ -A\cos\tau\cos\sigma \\ -A\sin\tau\cos\sigma \\ 0 \\ \vdots \end{pmatrix}, \quad X''^\mu = \begin{pmatrix} 0 \\ -A\cos\tau\cos\sigma \\ -A\sin\tau\cos\sigma \\ 0 \\ \vdots \end{pmatrix}$$

$$\ddot{X}^\mu - X''^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

still τ-dep.

b) Neumann: $\frac{\partial \mathcal{L}}{\partial X'^\mu} = 0 \;\hat{=}\; \frac{\partial}{\partial\sigma}\Big|_{b.c} = 0$, Dirichlet: $\delta X^\mu = 0 \;\hat{=}\; \frac{\partial}{\partial\tau}\Big|_{b.c} = 0$

$$\frac{\partial \mathcal{L}}{\partial X'^\mu}\Big|_{bound.} \sim X'_\mu\Big|_{bound.} = \begin{pmatrix} 0 \\ -A\cos\tau\sin\sigma \\ -A\sin\tau\sin\sigma \\ 0 \\ \vdots \end{pmatrix}\Bigg|_{\sigma = 0,\pi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

⇝ Neumann

For the velocity at the endpoints:

$$\frac{dx^1}{dx^0} = -\sin\tau\cos\sigma, \quad \frac{dx^2}{dx^0} = \cos\tau\cos\sigma, \quad \frac{dx^i}{dx^0} = 0, \; i = 3, \dots D$$

$$\leadsto \quad \vec{v}\Big|_{\sigma = 0,\pi} = \begin{pmatrix} \mp\sin\tau \\ \pm\cos\tau \\ 0 \\ \vdots \end{pmatrix} \quad \leadsto \quad \vec{v}^{\,2}\Big|_{\sigma = 0,\pi} = 1 \; (= c)$$

c) total energy $M = P^0 = \int_0^\pi p^0(\tau,\sigma)\, d\sigma = T \int_0^\pi d\sigma\, \dot{x}^0$

$$= A\pi T$$

d) $J = |J^{12}| = \left|\int_0^\pi d\sigma\, J^{12}\right| = \left|\int_0^\pi d\sigma \{x^1 p^2 - x^2 p^1\}\right|$

$p = T\dot{x}$ $$= \left|T\int_0^\pi d\sigma \{A^2 \cos^2\tau \cos^2\sigma + A^2 \sin^2\tau \cos^2\sigma\}\right|$$

$$= \left|A^2 T \int_0^\pi d\sigma \cos^2\sigma\right| = \frac{A^2 T\pi}{2}$$

Why only J^{12} important. → only space ang. mom., no J^{0i} etc., only boosts (not rotation)

e) $\alpha' = \frac{J}{M^2} = \frac{A^2 T\pi}{2A^2\pi^2 T^2} = \frac{1}{2\pi T}$

$\sqrt{-h}$ or $\sqrt{h}$? $\Rightarrow \sqrt{-h}$

And $\frac{\delta S}{\delta h^{\mu\nu}}$ or $\frac{\delta S}{\delta h_{\mu\nu}}$? $\Rightarrow$ up to match indices

3) $T_{\mu\nu} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\mu\nu}}$

for $S_P = -\frac{T}{2}\int_\Sigma d^2\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$

was calculated to be (see lecture) $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ σ, τ

two dim.

$$T_{\alpha\beta} = \frac{1}{2}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4}h_{\alpha\beta}\left(h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu\right) \overset{!}{=} 0$$

↑ factor 2 ?!

Changing to light-cone coordinates, $\sigma^\pm = \tau \pm \sigma$, and pulling the Minkowski metric onto the WS, i.e. (or WS metric in (τ,σ) coord. onto $\sigma^\pm$-WS ?!)

Is this really the pullback or transformed Mink. metric? $\Rightarrow$ No, just trafo of coord. doesn't have to be Pullback in general, so $h_{\alpha\beta}$ can be chosen arbitrarily $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$ then trafo to LCC $\begin{pmatrix}0&-1/2\\-1/2&0\end{pmatrix}$

$$\tau = \frac{1}{2}(\sigma^+ + \sigma^-), \quad \sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$$

$$h_{\alpha\beta} = \frac{\partial\sigma^\mu}{\partial\sigma'^\alpha}\frac{\partial\sigma^\nu}{\partial\sigma'^\beta}\eta_{\mu\nu} \Rightarrow \left.\begin{aligned} h_{++} &= 0 = h_{--} \\ h_{+-} &= -1/2 = h_{-+}\end{aligned}\right\} h_{\alpha\beta} = \begin{pmatrix}0&-1/2\\-1/2&0\end{pmatrix}$$

$$\Rightarrow h^{\alpha\beta} = \begin{pmatrix}0&-2\\-2&0\end{pmatrix}$$

We find
$$T_{++} = \frac{1}{2}\partial_+ X^\mu \partial_+ X_\mu$$
$$T_{--} = \frac{1}{2}\partial_- X^\mu \partial_- X_\mu$$
$$T_{+-} = \frac{1}{2}\partial_+ X^\mu \partial_- X_\mu + \frac{1}{8}\left(\partial_+ X^\mu \partial_- X_\mu \cdot (-2) + \partial X^\mu \partial_+ X_\mu \cdot (-2)\right)$$
$$= 0$$
$$= T_{-+}$$

Using $X^\mu(\sigma^+,\sigma^-) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu \sigma^- + \frac{i}{2}l_s \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}$$
$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu \sigma^+ + \frac{i}{2}l_s \sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}$$

we find
$$\partial_+ X^\mu = \frac{1}{2}l_s^2 p^\mu + l_s\sum_{n\neq 0}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}$$
$$p^\mu = \frac{2}{l_s}\tilde{\alpha}_0^\mu \Rightarrow \; = l_s \sum_n \tilde{\alpha}_n^\mu e^{-2in\sigma^+}$$
$$\partial_- X^\mu = l_s \sum_n \alpha_n^\mu e^{-2in\sigma^-}$$

$$\Rightarrow T_{++} - \partial_+ X^\mu \partial_+ X_\mu = l_s^2 \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\tilde{\alpha}_n^\mu \tilde{\alpha}_{m\mu} e^{-2i\sigma^+(n+m)}$$

$n+m = n'$
$$= l_s^2 \sum_{n'=\infty}^{\infty}\sum_{m=-\infty}^{\infty}\tilde{\alpha}_{n'-m}\cdot\tilde{\alpha}_m e^{-2i\sigma^+ n'}$$

$n' \to m$, $m \to n$

$$T_{--} = \partial_- X^\mu \partial_- X_\mu = \ell_s^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \alpha_n \cdot \alpha_m \, e^{-2i\sigma^-(n+m)}$$

$$\overset{n=n'-m}{=} \ell_s^2 \sum_{n'=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \alpha_{n'-m} \alpha_m \, e^{-2i\sigma^- n'}$$

$$\overset{\substack{n'\mapsto m \\ m\mapsto n}}{=} 2\ell_s^2 \sum_{m=-\infty}^{\infty} L_m e^{-2im\sigma^-} \, , \quad L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \alpha_n$$

We also find:

$$\{L_n, L_m\} = \frac{1}{4} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \{\alpha_{n-k} \cdot \alpha_k , \alpha_{m-l} \cdot \alpha_l\}$$

$$= \frac{1}{4} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left[\alpha^\mu_{n-k} \{\alpha_{k\mu}, \alpha_{m-l} \cdot \alpha_l\} + \{\alpha^\mu_{n-k}, \alpha_{m-l} \cdot \alpha_l\} \alpha_{k\mu} \right]$$

$$= \frac{1}{4} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \Big[\alpha^\mu_{n-k} \left(\alpha^\nu_{m-l} \{\alpha_{k\mu}, \alpha_{l\nu}\} + \{\alpha_{k\mu}, \alpha_{m-l\,\nu}\} \alpha_l^\nu \right)$$
$$+ \left(\alpha_{m-l\,\nu} \{\alpha^\mu_{n-k}, \alpha_l^\nu\} + \{\alpha^\mu_{n-k}, \alpha^\nu_{m-l}\} \alpha_{l\nu} \right) \alpha_{k\mu} \Big]$$

$$= \frac{1}{4} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \Big[\alpha^\mu_{n-k} \alpha^\nu_{m-l} \left(-ik \eta_{\mu\nu} \delta_{k+l,0} \right) + \alpha^\mu_{n-k} \alpha^\nu_l \left(-ik \eta_{\mu\nu} \delta_{k+m-l,0} \right)$$
$$+ \alpha_{m-l\,\nu} \alpha_{k\mu} \left(i l \eta^{\mu\nu} \delta_{n+l-k,0} \right) + \alpha_{l\nu} \alpha_{k\mu} \left(-i(n-k) \eta^{\mu\nu} \delta_{n+m-k-l,0} \right) \Big]$$

$$= \frac{-i}{4} \sum_{k=-\infty}^{\infty} \Big[\alpha_{n-k} \cdot \alpha_{m+k} \cdot k + \alpha_{n-k} \cdot \alpha_{k+m} \cdot k$$
$$- \alpha_{m+n-k} \cdot \alpha_k (k-n) + \alpha_{n+m-k} \cdot \alpha_k (n-k) \Big]$$

shift lower row of the sum by $k \mapsto k+m$

$$= \frac{-i}{4} \sum_{k=-\infty}^{\infty} \Big[\alpha_{n-k} \cdot \alpha_{m+k} \cdot k + \alpha_{n-k} \cdot \alpha_{k+m} \cdot k$$
$$- \alpha_{n-k} \alpha_{k+m} (m-n+k) + \alpha_{n-k} \alpha_{k+m} (n-m-k) \Big]$$

$$= -\frac{i}{4} \sum_{k=-\infty}^{\infty} \left[\alpha_{n-k} \alpha_{k+m} (n-m) + \alpha_{n-k} \alpha_{k+m} (n-m) \right]$$

shift $k \mapsto k-m$

$$= -\frac{i}{2} \sum_{k=-\infty}^{\infty} \alpha_{n+m-k} \alpha_k (n-m)$$

$$= -i(n-m) L_{n\cancel{+m}}$$

✓

sign wrong on sheet? yes

1. Having the Polyakov action

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

with the WS metric $h_{\alpha\beta} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ $\left(\rightsquigarrow h^{\alpha\beta} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)$

why not $h_{\alpha\beta}$ from before $h_{\alpha\beta} = \begin{pmatrix}0 & -1/2\\ -1/2 & 0\end{pmatrix}$? or insert here the given $h_{\alpha\beta}$ but w/ $\alpha,\beta \hat{=} +,-$? So $(\partial_+ X)^2 + (\partial_- X)^2$ $\rightsquigarrow$ then factor 2 less?

we find $$S_P = \frac{T}{2}\int_\Sigma d^2\sigma \left\{\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu\right\}$$

and with ? $\sigma^\pm = \tau \pm \sigma \rightsquigarrow \tau = \frac{1}{2}(\sigma^+ + \sigma^-)$

$\sigma = \frac{1}{2}(\sigma^+ - \sigma^-)$

$\partial_+ = \frac{1}{2}(\partial_\tau + \partial_\sigma)\,,\quad \partial_\tau = \partial_+ + \partial_-$

$\partial_- = \frac{1}{2}(\partial_\tau - \partial_\sigma)\,,\quad \partial_\sigma = \partial_+ - \partial_-$

$$S_P = \frac{T}{2}\int_\Sigma d^2\sigma \left\{[(\partial_+ + \partial_-)X]^2 - [(\partial_+ - \partial_-)X]^2\right\}$$

$$= \frac{T}{2}\int_\Sigma d^2\sigma \left\{4\partial_+ X^\mu \partial_- X_\mu\right\} = 2T\int_\Sigma d^2\sigma \left\{\partial_+ X^\mu \partial_- X_\mu\right\}$$

it's τ,σ, not $\sigma^\pm$

With $\delta X^\mu = a_n e^{2in\sigma^-}\partial_- X^\mu$, S_P then transforms as

$$S_P \longrightarrow 2T\int_\Sigma d^2\sigma \left\{\partial_+(X^\mu + a_n e^{2in\sigma^-}\partial_- X^\mu)\partial_-(X_\mu + a_n e^{2in\sigma^-}\partial_- X_\mu)\right\}$$

$$\overset{\substack{\partial_+\partial_- X^\mu = 0\\ \text{e.o.m.}}}{=} S_P + 2T\int_\Sigma d^2\sigma \left\{\partial_+ X^\mu \partial_-(a_n e^{2in\sigma^-}\partial_- X_\mu)\right\}$$

$$\overset{\partial_+\partial_- X^\mu = 0}{=} S_P + 2T\int_\Sigma d^2\sigma\, \partial_-\left\{(\partial_+ X^\mu) a_n e^{2in\sigma^-}\partial_- X_\mu\right\}$$

$$\rightsquigarrow \delta\mathcal{L} = \mathcal{L} + \partial_\alpha F^\alpha$$

$$j_n^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha X^\mu)}\delta X_n^\mu - F^\alpha \rightsquigarrow j^- = 0$$

j^+ as on next page!

$d\sigma^+ = d\sigma \frac{d\sigma^+}{d\sigma}$

$$-\frac{T}{2}\int d^2\sigma^\pm \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}_{+,-}\partial_\alpha X^\mu \partial_\beta X_\mu$$

$$= +\frac{T}{2}\int d^2\sigma^\pm\, \partial_+ X^\mu \partial_- X_\mu \cdot 2 = T\int d^2\sigma^\pm\, \partial_+ X^\mu \partial_- X_\mu$$

$= T\int d^2\sigma\, \partial_+ X^\mu \partial_- X_\mu$

2. With $\mathcal{L} = 2T\partial_+ X^\mu \partial_- X_\mu$, the Noether current is given by

$$\epsilon^i j_i^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^a)}\delta X^a$$ and for the 0^{th} component thus

$$\epsilon^i j_i^0 = 2T(\partial_- X_\mu)(a_n e^{2in\sigma^-}\partial_- X^\mu)$$

$\hat{=} n$ on r.h.s.

$$= 2T(\partial_- X_\mu)(\partial_- X^\mu)\, a_n e^{2in\sigma^-}$$

$$\Rightarrow j_n^0 = T(\partial_- X_\mu)(\partial_- X^\mu)e^{2in\sigma^-}$$

−1 There are j^+, j^- this is j^+.

✓ why not consider j^{1}? vanishes? $\Rightarrow j^- = 0$, $j^+ \neq 0$ here

3. The charge is defined as

$$Q_n = \int d\sigma\, j^0_{\,j^+} = T\int d\sigma\, (\partial_- X^\mu)(\partial_- X_\mu)e^{2in\sigma^-}$$

$$\overset{\text{eq (18)}}{=} T\int d\sigma\, T_{--}\, e^{2in\sigma^-}$$

$$\overset{\text{eq (20)}}{=} T\int d\sigma\, 2l_s^2 \sum_{m=-\infty}^{\infty} L_m e^{2i(\tau-\sigma)(n-m)}$$

vanishes for $n \neq m$ $= 2Tl_s^2\pi L_n$

$T = \frac{1}{2\pi l_s^2} \Rightarrow = L_n$

✓ σ^- or σ integral $\Rightarrow \sigma$ because it's the spatial int.

? why now $T = \frac{1}{2\pi l_s^2}$?