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String Theory Exercise 3 Homework

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5+10+10

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1) **1.1** WS coordinates $\tau \in [-\infty, \infty]$, $\sigma \in [0, \pi]$

Mode expansion of bosonic string: $X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma)$

only for closed string?

$$X^\mu(\sigma,\tau) = x^\mu + l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}\right)$$

$$\xrightarrow[\tau \mapsto -i\tau]{\text{Wick rot.}} x^\mu - i l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-2in(-i\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(-i\tau+\sigma)}\right)$$

$$= x^\mu - i l_s^2 p^\mu \tau + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-2n(\tau-i\sigma)} + \tilde{\alpha}_n^\mu e^{-2n(\tau+i\sigma)}\right)$$

$\tau \mapsto -i\tau$ "metric on WS becomes euclidean" – but no metric here; why okay to just do this trafo w/o integral? ⇝ 2d scalar field theory; same reason as in QFT $(\partial_x^2 + \partial_y^2 + m^2) G(x,y) = \delta^{(2)}(x-y)$

$$\left| \; \zeta = 2(\tau - i\sigma), \quad \bar{\zeta} = 2(\tau + i\sigma) \rightsquigarrow \tau = \frac{1}{4}(\zeta + \bar{\zeta}) \right.$$

$$\left| \; z = e^{\zeta} = e^{2(\tau - i\sigma)} \quad \bar{z} = e^{\bar{\zeta}} = e^{2(\tau+i\sigma)} \rightsquigarrow |z|^2 = e^{4\tau} = e^{\zeta+\bar{\zeta}} \rightsquigarrow \zeta + \bar{\zeta} = \log(|z|^2) \right.$$

$$= x^\mu - i\frac{l_s^2}{4} p^\mu \log(|z|^2) + \frac{i}{2} l_s \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu z^{-n} + \tilde{\alpha}_n^\mu \bar{z}^{-n}\right)$$

Why N.O. for $:xp:$? Because $[x,p] \neq 0$? And same argument s.t. $:\alpha_n \tilde{\alpha}_m: = \alpha_n \tilde{\alpha}_m$ because $[\alpha_n, \tilde{\alpha}_m] = 0$ and $:\alpha_n x:$ etc.? why not $:xp: = px$? ⇝ yes

Same x^μ in both X^μ? ⇝ yes, same function

Spacetime index has to be the same because otherwise N.O. has no effect for flat metric $\eta^{\mu\nu}$? ⇝ yes; fixed the gauge and have flat metric

1.2 We note that we only have to consider such terms explicitly, where $[\cdot,\cdot] \neq 0$, because then after normal ordering and using the commutator, we get additional terms compared to the not normal ordered one. Otherwise, the normal ordering has no effect.

We thus find:

$$:X(z,\bar{z})X(w,\bar{w}): = X(z,\bar{z})X(w,\bar{w})$$

$$\underbrace{+ \frac{l_s^2}{4}\left\{\sum_{n\neq 0} \frac{1}{n}(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n})\right\}\left\{\sum_{m\neq 0} \frac{1}{m}(\alpha_m w^{-m} + \tilde{\alpha}_m \bar{w}^{-m})\right\} - \frac{l_s^2}{4} :\left\{\sum_{n\neq 0} \frac{1}{n}(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n})\right\}\left\{\sum_{m\neq 0} \frac{1}{m}(\alpha_m w^{-m} + \tilde{\alpha}_m \bar{w}^{-m})\right\}:}_{(1)}$$

$$\underbrace{+ i\frac{l_s^2}{4} xp \log(|w|^2) + i\frac{l_s^2}{4} px \log(|z|^2) - i\frac{l_s^2}{4} :xp: \log(|w|^2) - i\frac{l_s^2}{4} :px: \log(|z|^2)}_{(2)}$$

where the spacetime indices have to be equal (see commutators, the flat metric $\eta^{\mu\nu}$ takes care of this).

Consider (1):

$$:\sum_{n,m\neq 0} \frac{1}{n\cdot m}\left(\alpha_n z^{-n} + \bar{\alpha}_n \bar{z}^{-n}\right)\left(\alpha_m w^{-m} + \bar{\alpha}_m \bar{w}^{-m}\right):$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{n\cdot m}\Big\{ \underbrace{:\alpha_n \alpha_m: z^{-n} w^{-m}} + \underbrace{:\alpha_n \bar{\alpha}_m: z^{-n}\bar{w}^{-m}}_{\text{no effect after commuting}} + \underbrace{:\bar{\alpha}_n \alpha_m: \bar{z}^{-n} w^{-m}} + \underbrace{:\bar{\alpha}_n \bar{\alpha}_m: \bar{z}^{-n}\bar{w}^{-m}}_{(*)}\Big\}$$

(*) Actually tr($\eta^{\mu\nu}$) for $\alpha_m \cdot \alpha_n$? ⇝ only one component

Problem if $\eta^{\mu\nu}$ not flat ⇝ $\alpha_m^\mu \alpha_n^\nu = \alpha_n^\nu \alpha_m^\mu + m\eta^{\mu\nu}\delta$ thus index μ,ν changes? ⇝ only equal indices (fixed the gauge)

$$\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{n\cdot m}\alpha_n \alpha_m z^{-n} w^{-m} + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{n\cdot m}\alpha_m \alpha_n z^{-n} w^{-m}$$

$$\Big|\ [\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{n\cdot m}\alpha_n^{\mu\, \nu/\mu}\alpha_m z^{-n} w^{-m} + \sum_{n=1}^{\infty}\frac{1}{n} z^{-n} w^{n}$$

Analogously $(*) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{n\cdot m}\bar{\alpha}_n \bar{\alpha}_m \bar{z}^{-n}\bar{w}^{-m} + \sum_{n=1}^{\infty}\frac{1}{n}\bar{z}^{-n}\bar{w}^{n}$

What if $:\alpha_m \alpha_n:$ for both $m,n>0$? Doesn't matter? ⇝ yes, commutator vanishes

⇝ The remaining part from (1) and the not-normal ordered one is

$$-\frac{\ell_s^2}{4}\left\{\sum_{n=1}^{\infty}\frac{1}{n} z^{-n} w^{n} + \sum_{n=1}^{\infty}\frac{1}{n}\bar{z}^{-n}\bar{w}^{n}\right\} = +\frac{\ell_s^2}{4}\left\{\log\left(1-\frac{w}{z}\right) + \log\left(1-\frac{w^*}{z^*}\right)\right\}$$

$$= \frac{\ell_s^2}{4}\log\left(1 - \frac{w}{z} - \frac{w^*}{z^*} + \frac{|w|^2}{|z|^2}\right)$$

Consider (2):

$$i\frac{\ell_s^2}{4}(px - xp)\log(|z|^2) \overset{[x^\mu, p^\nu] = i\eta^{\mu\nu}}{=} \frac{\ell_s^2}{4}\log(|z|^2)$$

$$⇝ :X(z,\bar{z})X(w,\bar{w}): = X(z,\bar{z})X(w,\bar{w}) + \frac{\ell_s^2}{4}\left\{\log\left(1 - \frac{w}{z} - \frac{w^*}{z^*} + \frac{|w|^2}{|z|^2}\right) + \log(|z|^2)\right\}$$

$$= X(z,\bar{z})X(w,\bar{w}) + \frac{\ell_s^2}{4}\log\left(|z|^2 - wz^* - w^*z + |w|^2\right)$$

$$⇝ X(z,\bar{z})X(w,\bar{w}) = :X(z,\bar{z})X(w,\bar{w}): - \frac{\ell_s^2}{4}\log\left(|z-w|^2\right)$$

2) $L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\alpha_n: \quad , \quad L_{m+n} = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha_{m+n-k}\alpha_k:$

$$[L_m, L_n] = \frac{1}{2}\sum_{k=-\infty}^{\infty}[:\alpha_{m-k}\alpha_k:, L_n]$$

$$= \frac{1}{2}\left\{\sum_{k=-\infty}^{0}[\alpha_k\alpha_{m-k}, L_n] + \sum_{k=1}^{\infty}[\alpha_{m-k}\alpha_k, L_n]\right\}$$

$$= \frac{1}{2}\left\{\sum_{k=-\infty}^{0}\left(\alpha_k[\alpha_{m-k}, L_n] + [\alpha_k, L_n]\alpha_{m-k}\right) + \sum_{k=1}^{\infty}\left(\alpha_{m-k}[\alpha_k, L_n] + [\alpha_{m-k}, L_n]\alpha_k\right)\right\}$$

$$[\alpha_i^\mu, L_n] = \frac{1}{2}\left[\alpha_i^\mu, \sum_{\ell=-\infty}^{\infty} :\alpha_{n-\ell}\alpha_\ell:\right]$$

$$= \frac{1}{2}\left\{\sum_{\ell=-\infty}^{0}[\alpha_i^\mu, \alpha_\ell\alpha_{n-\ell}] + \sum_{\ell=1}^{\infty}[\alpha_i^\mu, \alpha_{n-\ell}\alpha_\ell]\right\}$$

$$= \frac{1}{2}\left\{\sum_{\ell=-\infty}^{0}\left(\alpha_{\ell\nu}[\alpha_i^\mu, \alpha_{n-\ell}^\nu] + [\alpha_i^\mu, \alpha_\ell^\nu]\alpha_{n-\ell\,\nu}\right) + \sum_{\ell=1}^{\infty}\left(\alpha_{n-\ell\,\nu}[\alpha_i^\mu, \alpha_\ell^\nu] + [\alpha_i^\mu, \alpha_{n-\ell}^\nu]\alpha_{\ell\nu}\right)\right\}$$

$$= \frac{1}{2}\left\{\sum_{\ell=-\infty}^{0} i\alpha_\ell^\mu\delta_{i+n-\ell,0} + i\alpha_{n-\ell}^\mu\delta_{i+\ell,0} + \sum_{\ell=1}^{\infty} i\alpha_{n-\ell}^\mu\delta_{i+\ell,0} + i\alpha_\ell^\mu\delta_{i+n-\ell,0}\right\}$$

$$= \frac{1}{2}i\left\{\alpha_{i+n}^\mu + \alpha_{i+n}^\mu\right\} = i\alpha_{i+n}^\mu$$

Where is the problem to use comm. at this point and get to form L_{m+n} w/ add. δ_{m+n} term? (even w/ [illegible] N.O.? No shifting in N.O. sum?)

$$= \frac{1}{2}\left\{\sum_{k=-\infty}^{0}\left((m-k)\alpha_k\alpha_{m-k+n} + k\alpha_{k+n}\alpha_{m-k}\right) + \sum_{k=1}^{\infty}\left(k\alpha_{m-k}\alpha_{k+n} + (m-k)\alpha_{m-k+n}\alpha_k\right)\right\} \quad (1)$$

$$= \frac{1}{2}\sum_k (m-k) :\alpha_{m+n-k}\alpha_k: + \underbrace{\frac{1}{2}\sum_{k=-\infty}^{0} k\alpha_{k+n}\alpha_{m-k} + \frac{1}{2}\sum_{k=1}^{\infty} k\alpha_{m-k}\alpha_{k+n}}_{(*)}$$

What if $n=0$ we can't split the sum again?

$$(*) \overset{k'=k+n}{=} \underbrace{\frac{1}{2}\sum_{k'=-\infty}^{n}(k'-n)\alpha_{k'}\alpha_{m+n-k'}}_{(**)} + \frac{1}{2}\sum_{k'=n+1}^{\infty}(k'-n)\alpha_{m+n-k'}\alpha_{k'}$$

$$(**) = \frac{1}{2}\sum_{k=-\infty}^{0}(k-n)\alpha_k\alpha_{m+n-k} + \frac{1}{2}\sum_{k=0}^{n}(k-n)\alpha_k\alpha_{m+n-k}$$

$$= \frac{1}{2}\sum_{k=-\infty}^{0}(k-n)\alpha_k\alpha_{m+n-k} + \frac{1}{2}\sum_{k=0}^{n}(k-n)\big(\alpha_{m+n-k}\alpha_k + k\delta_{m+n,0}\overset{\text{ST dim.}}{D}\big)$$

$\eta_{\mu\nu}\eta^{\mu\nu} = D$ $\mathrm{tr}(\eta)$? ⇝ definition of inverse metric that $\eta_{\mu\nu}\eta^{\nu\kappa} = \delta_\mu^\kappa$ yields a sum of "one's".

$$= \frac{1}{2}\sum_{k=-\infty}^{0}(k-n)\alpha_k\alpha_{m+n-k} + \frac{1}{2}\sum_{k=0}^{n}(k-n)\alpha_{m+n-k}\alpha_k + \frac{1}{2}\sum_{k=n+1}^{\infty}(k-n)\alpha_{m+n-k}\alpha_k$$

$$+\frac{1}{2}\sum_{k=0}^{n}(k-n)k\,\delta_{m+n,0}D$$

$$= \frac{1}{2}\sum_{k}(k-n):\alpha_{m+n-k}\alpha_k: + \frac{D}{2}\sum_{k=1}^{n}(k-n)k\,\delta_{m+n,0}$$

$$= \frac{1}{2}(m-n)\sum_{k}:\alpha_{m+n-k}\alpha_k: + \frac{D}{2}\sum_{k=1}^{n}(k-n)k\,\delta_{m+n,0}$$

$$= (m-n)L_{m+n} + \delta_{m+n,0}\frac{D}{2}\Big(\underbrace{\underbrace{\sum_{k=1}^{n}k^2}_{\frac{n(n+1)(2n+1)}{6}} - n\underbrace{\sum_{k=1}^{n}k}_{\frac{n(n+1)}{2}}}_{\frac{(n^2+n)(2n+1)-3n^3-3n^2}{6} = \frac{-n^3+n}{6}}\Big)$$

N.O. effects appear when $m+n=0$ — but also for the rest N.O.? ⇝ yes, but no effect

$$= (m-n)L_{m+n} + \delta_{m+n,0}\frac{D}{12}(m^3 - \cancel{m})$$

Alternatively, we could have stopped at (1), commute the operators and shift the sums, in order to see that $[L_m, L_n] = (m-n)L_{m+n} + T\delta_{m+n,0}$ with T being some number.

We then make the Ansatz: $[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}$ and notice: $[L_m, L_{-m}] = 2mL_0 + A(m) = -[L_{-m}, L_m]$

$[L_{-m}, L_m] = -2mL_0 + A(-m)$

⇝ $A(-m) = -A(m)$

$[X, c] = 0$ clear? ⇝ see below

$[L_k, A(m)]$ (per def.)

$A(m)$ could also be infinite! ⇝ for scalar field $= 1$ in 2 dimensions

Also, from the Jacobi identity, we find

$$0 \overset{!}{=} [L_k,[L_n,L_m]] + [L_n,[L_m,L_k]] + [L_m,[L_k,L_n]]$$

$$= [L_k, (n-m)L_{n+m} + A(n)\delta_{n+m}] + [L_n, (m-k)L_{m+k} + A(m)\delta_{m+k}]$$

$$+ [L_m, (k-n)L_{k+n} + A(k)\delta_{k+n}]$$

$[L_k, A(m)] = 0$? ⇝ it's a c-number per def. commutes

$$= (n-m)\{(k-n-m)L_{k+n+m} + A(k)\delta_{k+n+m}\} + (m-k)\{(n-m-k)L_{k+n+m} + A(n)\delta_{n+m+k}\}$$

$$+ (k-n)\{(m-k-n)L_{k+n+m} + A(m)\delta_{m+k+n}\}$$

$$= \{(n-m)A(k) + (m-k)A(n) + (k-n)A(m)\}\delta_{k+n+m} \quad (*)$$

What about δ_{m+n} ⇝ choosing values which

Using the values $k=1$, $m=-n-1$ ($\Rightarrow k+n+m=0$), we find

$$(2n+1)A(1) + (-n-2)A(n) + (1-n)A(-n-1) = 0$$

$$\Leftrightarrow \quad A(n+1)(1-n) = (2n+1)A(1) - (n+2)A(n)$$

$$\Leftrightarrow \quad A(n+1) = \frac{(n+2)A(n) - (2n+1)A(1)}{n-1}$$

Where from Ansatz? $\Rightarrow$ has to be odd in m and can see from (**) that higher terms drop out.

Trying the Ansatz $A(n) = C_3 n^3 + C_1 n$, we find

$$A(n+1) = C_3(n+1)^3 + C_1(n+1)$$

$$\overset{!}{=} \frac{(n+2)(C_3 n^3 + C_1 n) - (2n+1)(C_3 + C_1)}{n-1}$$

$$= \frac{C_3 n^4 + C_1 n^2 + 2C_3 n^3 + \cancel{2C_1 n} - 2C_3 n - \cancel{2C_1 n} - C_3 - C_1}{n-1}$$

$$= \frac{C_3(n^4 + 2n^3 - 2n - 1) + C_1(n^2-1)}{n-1}$$

$$= \frac{C_3(n-1)(n^3+3n^2+3n+1) + C_1(n-1)(n+1)}{n-1}$$

$$= C_3(n+1)^3 + C_1(n+1) \checkmark$$

To determine C_1 and C_3, we look at $[L_{1/2}, L_{-1/2}]$ on the vac. state

Why only valid for the $p=0$ state? $\Rightarrow$ can transform it like this by Lorentz Trafo

no momentum!

$$\langle 0,0|[L_1, L_{-1}]|0,0\rangle = \frac{1}{4}\Big\{\langle 0,0|\sum_n :\alpha_{1-n}\alpha_n: \sum_m :\alpha_{-1-m}\alpha_m:|0,0\rangle$$

$$- \langle 0,0|\sum_n :\alpha_{-1-n}\alpha_n: \sum_m :\alpha_{1-m}\alpha_m:|0,0\rangle\Big\}$$

from normal ordering, both have to be creation (acting to right) and/or annihilation (acting to right) operators s.t. the vac. does not get annihilated.

i.e. $m<0 \wedge -1-m<0$ ↯
$n>0 \wedge 1-n>0$ ↯
$m<0 \wedge 1-m<0$ ↯
$n>0 \wedge -1-n>0$ ↯

here $n,m=0$ not allowed, as $\alpha_0 \sim p$ and $p=0$ for our vac.

$= 0$

Also $\langle 0,0|[L_1, L_{-1}]|0,0\rangle = \langle 0,0| 2L_0 + A(1)|0,0\rangle$

$= \underbrace{\langle 0,0|\alpha_0\alpha_0|0,0\rangle}_{=0 \text{ for } p\sim 0} + C_1 + C_3$

$\Rightarrow C_1 = -C_3$

Analogously: $\langle 0,0|[L_2, L_{-2}]|0,0\rangle = \langle 0,0|4L_0 + A(2)|0,0\rangle$

$= 2C_1 + 8C_3$

and $\langle 0,0|[L_2,L_{-2}]|0,0\rangle = \frac{1}{4}\{\langle 0,0|\sum_n :\alpha_{2-n}\alpha_n: \sum_m :\alpha_{-2-m}\alpha_m:|0,0\rangle$

$- \langle 0,0|\sum_n :\alpha_{-2-n}\alpha_n: \sum_m :\alpha_{2-m}\alpha_m:|0,0\rangle\}$

$\Rightarrow$ conditions: $m<0 \wedge -2-m<0 \Rightarrow m=-1$

$n>0 \wedge 2-n>0 \Rightarrow n=1$

$m<0 \wedge 2-m<0 \Rightarrow \nexists$

$n>0 \wedge -2-n>0 \Rightarrow \nexists$

$= \frac{1}{4}\langle 0,0|\alpha_1\cdot\alpha_1\,\alpha_{-1}\cdot\alpha_{-1}|0,0\rangle = \frac{1}{4}\langle 0,0|\alpha_1^\mu\alpha_1^\nu\eta_{\mu\nu}\eta_{\kappa\rho}\alpha_{-1}^\kappa\alpha_{-1}^\rho|0,0\rangle$

$= \frac{1}{4}\eta_{\mu\nu}\eta_{\kappa\rho}\langle 0,0|\alpha_1^\mu\alpha_1^\nu\alpha_{-1}^\kappa\alpha_{-1}^\rho|0,0\rangle$

$[\alpha_1^\mu\alpha_1^\nu, \alpha_{-1}^\kappa\alpha_{-1}^\rho] = \alpha_1^\mu[\alpha_1^\nu, \alpha_{-1}^\kappa\alpha_{-1}^\rho] + [\alpha_1^\mu, \alpha_{-1}^\kappa\alpha_{-1}^\rho]\alpha_1^\nu$

$= \alpha_1^\mu\alpha_{-1}^\kappa[\alpha_1^\nu, \alpha_{-1}^\rho] + \alpha_1^\mu[\alpha_1^\nu, \alpha_{-1}^\kappa]\alpha_{-1}^\rho$

$+ \alpha_{-1}^\kappa[\alpha_1^\mu, \alpha_{-1}^\rho]\alpha_1^\nu + [\alpha_1^\mu, \alpha_{-1}^\kappa]\alpha_{-1}^\rho\alpha_1^\nu$

$= \eta^{\nu\rho}\alpha_1^\mu\alpha_{-1}^\kappa + \eta^{\nu\kappa}\alpha_1^\mu\alpha_{-1}^\rho + \eta^{\mu\rho}\alpha_{-1}^\kappa\alpha_1^\nu + \eta^{\mu\kappa}\alpha_{-1}^\rho\alpha_1^\nu$

$= \frac{1}{4}\eta_{\mu\nu}\eta_{\kappa\rho}\{\langle 0,0|\alpha_{-1}^\kappa\alpha_{-1}^\rho\alpha_1^\mu\alpha_1^\nu + \eta^{\nu\rho}\alpha_1^\mu\alpha_{-1}^\kappa + \eta^{\nu\kappa}\alpha_1^\mu\alpha_{-1}^\rho + \eta^{\mu\rho}\alpha_{-1}^\kappa\alpha_1^\nu + \eta^{\mu\kappa}\alpha_{-1}^\rho\alpha_1^\nu|0,0\rangle\}$

$m\geq 0 \Rightarrow$ annihilation of vac

$= \frac{1}{4}\eta_{\mu\nu}\eta_{\kappa\rho}\langle 0,0|\eta^{\nu\rho}\alpha_1^\mu\alpha_{-1}^\kappa + \eta^{\nu\kappa}\alpha_1^\mu\alpha_{-1}^\rho|0,0\rangle$

$= \frac{1}{4}\eta_{\mu\nu}\eta_{\kappa\rho}\{\eta^{\nu\rho}\langle 0,0|\alpha_{-1}^\kappa\alpha_1^\mu + \eta^{\mu\kappa}|0,0\rangle + \eta^{\nu\kappa}\langle 0,0|\alpha_{-1}^\rho\alpha_1^\mu + \eta^{\mu\rho}|0,0\rangle\}$

$= \frac{1}{4}\{\eta_{\mu\nu}\eta_{\kappa\rho}\eta^{\nu\rho}\eta^{\mu\kappa} + \eta_{\mu\nu}\eta_{\kappa\rho}\eta^{\nu\kappa}\eta^{\mu\rho}\} = \frac{D}{2}$

$\Rightarrow 2C_1 + 8C_3 = 6C_3 = \frac{D}{2}, \Rightarrow C_3 = \frac{D}{12} \Rightarrow C_1 = -\frac{D}{12}$

$\Rightarrow A(m) = \frac{D}{12}(m^3 - m)$

3) Had $\mathcal{J}^{\mu\nu} = \int d\sigma\, J^{\mu\nu} = \ell^{\mu\nu} + E^{\mu\nu} + \bar{E}^{\mu\nu}$

Same w/ N.O. as W/O. $\qquad$ $:x^\mu p^\nu - x^\nu p^\mu:$ $\qquad$ $-i\sum_{n>0} \frac{1}{n}\left(:\alpha_{-n}^{\mu}\alpha_n^{\nu}: - :\alpha_{-n}^{\nu}\alpha_n^{\mu}:\right)$

✓

What about L_m^2? (containing $\sum_n \alpha_{m-n}\alpha_n$) ⇝ same result, as zero-modes of them equal

W/O N.O. never calculated this? trivially vanishes? ⇝ does not make sense, as we look at quantized theory

With $L_m = \frac{1}{2}\sum_n :\alpha_{m-n}\alpha_n:$, we notice that in $[L_m, \mathcal{J}^{\mu\nu}]$ only commutators between the α's and α/xp contribute, i.e. $[L_m, \mathcal{J}^{\mu\nu}] = [L_m, \ell^{\mu\nu} + E^{\mu\nu}]$.

$$[L_m, \mathcal{J}^{\mu\nu}] \supset [L_m, E^{\mu\nu}] = -\frac{i}{2}\sum_{n=-\infty}^{\infty}\sum_{k>0}\frac{1}{k}\left[:\alpha_{m-n}\alpha_n:, :\alpha_{-k}^{\mu}\alpha_k^{\nu}: - :\alpha_{-k}^{\nu}\alpha_k^{\mu}:\right]$$

$$= -\frac{i}{2}\Big\{\sum_{k>0}\frac{1}{k}\sum_{n=-\infty}^{0}\left[\alpha_n\alpha_{m-n}, \alpha_{-k}^{\mu}\alpha_k^{\nu} - \alpha_{-k}^{\nu}\alpha_k^{\mu}\right]$$
$$+ \sum_{k>0}\frac{1}{k}\sum_{n=1}^{\infty}\left[\alpha_{m-n}\alpha_n, \alpha_{-k}^{\mu}\alpha_k^{\nu} - \alpha_{-k}^{\nu}\alpha_k^{\mu}\right]\Big\}$$

$$[\alpha_m^{\mu}, \alpha_n^{\nu}] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$
$$\leadsto \alpha_{m-n}^{\mu}\alpha_n^{\nu}\eta_{\mu\nu} = \eta_{\mu\nu}\{\alpha_n^{\nu}\alpha_{m-n}^{\mu} + (m-n)\,\delta_{m,0}\,\eta^{\mu\nu}\}$$

$$= -\frac{i}{2}\Big\{\sum_{k>0}\frac{1}{k}\sum_{n=-\infty}^{\infty}\left[\alpha_n\alpha_{m-n}, \alpha_{-k}^{\mu}\alpha_k^{\nu} - \alpha_{-k}^{\nu}\alpha_k^{\mu}\right]$$
$$+ \sum_{k>0}\frac{1}{k}\sum_{n=1}^{\infty}\underbrace{\left[D(m-n)\delta_{m,0}, \alpha_{-k}^{\mu}\alpha_k^{\nu} - \alpha_{-k}^{\nu}\alpha_k^{\mu}\right]}_{=0}\Big\}$$

$$= -\frac{i}{2}\sum_{k>0}\frac{1}{k}\sum_{n=-\infty}^{\infty}\Big\{\alpha_{n\kappa}\left[\alpha_{m-n}^{\kappa}, \alpha_{-k}^{\mu}\alpha_k^{\nu}\right] + \left[\alpha_n^{\kappa}, \alpha_{-k}^{\mu}\alpha_k^{\nu}\right]\alpha_{m-n\,\kappa}$$
$$- \alpha_{n\kappa}\left[\alpha_{m-n}^{\kappa}, \alpha_{-k}^{\nu}\alpha_k^{\mu}\right] - \left[\alpha_n^{\kappa}, \alpha_{-k}^{\nu}\alpha_k^{\mu}\right]\alpha_{m-n\,\kappa}\Big\}$$

$$= -\frac{i}{2}\sum_{k>0}\frac{1}{k}\sum_{n=-\infty}^{\infty}\Big\{(m-n)\,\delta_{m-n+k}\,\eta^{\kappa\nu}\alpha_{n\kappa}\alpha_{-k}^{\mu} + (m-n)\delta_{m-n-k}\eta^{\kappa\mu}\alpha_{n\kappa}\alpha_k^{\nu}$$
$$+ n\,\delta_{n+k}\,\eta^{\kappa\nu}\alpha_{-k}^{\mu}\alpha_{m-n\,\kappa} + n\,\delta_{n-k}\,\eta^{\kappa\mu}\alpha_k^{\nu}\alpha_{m-n\,\kappa}$$
$$- (m-n)\,\delta_{m-n+k}\,\eta^{\kappa\mu}\alpha_{n\kappa}\alpha_{-k}^{\nu} - (m-n)\,\delta_{m-n-k}\,\eta^{\kappa\nu}\alpha_{n\kappa}\alpha_k^{\mu}$$
$$- n\,\delta_{n+k}\,\eta^{\kappa\mu}\alpha_{-k}^{\nu}\alpha_{m-n\,\kappa} - n\,\delta_{n-k}\,\eta^{\kappa\nu}\alpha_k^{\mu}\alpha_{m-n\,\kappa}\Big\}$$

$$= -\frac{i}{2}\sum_{k>0}\frac{1}{\cancel{k}}\Big\{(-\cancel{k})\,\alpha_{m+k}^{\nu}\alpha_{-k}^{\mu} + \cancel{k}\,\alpha_{m-k}^{\mu}\alpha_k^{\nu} - \cancel{k}\,\alpha_{-k}^{\mu}\alpha_{m+k}^{\nu} + \cancel{k}\,\alpha_k^{\nu}\alpha_{m-k}^{\mu}$$
$$+ \cancel{k}\,\alpha_{m+k}^{\mu}\alpha_{-k}^{\nu} - \cancel{k}\,\alpha_{m-k}^{\nu}\alpha_k^{\mu} + \cancel{k}\,\alpha_{-k}^{\nu}\alpha_{m+k}^{\mu} - \cancel{k}\,\alpha_k^{\mu}\alpha_{m-k}^{\nu}\Big\}$$

The underlined terms correspond to each other respectively with $k \mapsto -k$, thus leaving the sum over all $k \neq 0$

$$= -\frac{i}{2} \sum_{k\neq 0} \left\{ -\alpha_{m+k}^{\nu} \alpha_{-k}^{\mu} + \alpha_{m+k}^{\mu} \alpha_{-k}^{\nu} - \alpha_k^{\mu} \alpha_{m-k}^{\nu} + \alpha_k^{\nu} \alpha_{m-k}^{\mu} \right\}$$

include $k=0$ term, and subtract

$$= -\frac{i}{2} \sum_{k=-\infty}^{\infty} \left\{ -\alpha_{m+k}^{\nu} \alpha_{-k}^{\mu} + \alpha_{m+k}^{\mu} \alpha_{-k}^{\nu} \overbrace{- \alpha_k^{\mu} \alpha_{m-k}^{\nu} + \alpha_k^{\nu} \alpha_{m-k}^{\mu}}^{(*)} \right\}$$

$$+ \frac{i}{2} \left\{ -\alpha_m^{\nu} \alpha_0^{\mu} + \alpha_m^{\mu} \alpha_0^{\nu} - \alpha_0^{\mu} \alpha_m^{\nu} + \alpha_0^{\nu} \alpha_m^{\mu} \right\}$$

$k \mapsto k+m$ in $(*)$

$$= -\frac{i}{2} \sum_{k=-\infty}^{\infty} \left\{ -\alpha_{m+k}^{\nu} \alpha_{-k}^{\mu} + \alpha_{m+k}^{\mu} \alpha_{-k}^{\nu} - \alpha_{k+m}^{\mu} \alpha_{-k}^{\nu} + \alpha_{k+m}^{\nu} \alpha_{-k}^{\mu} \right\}$$

$[\alpha_0^{\mu}, \alpha_m^{\nu}] = 0 \cdot \delta_{m,0} \eta^{\mu\nu} = 0$

$$+ i \left\{ -\alpha_0^{\mu} \alpha_m^{\nu} + \alpha_m^{\mu} \alpha_0^{\nu} \right\}$$

$$= i \left\{ \alpha_m^{\mu} \alpha_0^{\nu} - \alpha_0^{\mu} \alpha_m^{\nu} \right\} = \underline{\underline{i \left\{ \alpha_m^{\mu} \alpha_0^{\nu} - \alpha_m^{\nu} \alpha_0^{\mu} \right\}}}$$

✓

Why do $\alpha_0^{\mu} \alpha_0^{\nu}$ still commute? because of the $0 \cdot$ ... or because $\alpha_0 \sim p$

We also find: $[L_m, J^{\mu\nu}] \supset [L_m, \ell^{\mu\nu}] = \frac{1}{2} \left[\sum_n :\alpha_{m-n} \alpha_n:, :x^{\mu} p^{\nu} - x^{\nu} p^{\mu}: \right]$

any non-commuting for an α_0 in the sum, i.e. $n=0, m$

$$= \frac{1}{2} \left[\alpha_m \alpha_0 + \alpha_0 \alpha_m, x^{\mu} p^{\nu} - x^{\nu} p^{\mu} \right]$$

as $p \sim \alpha_0$ and $p \sim \alpha_0$ commutes w/ all α but $[p, x] \neq 0$

$$[\alpha_m \alpha_n, x^{\mu} p^{\nu}] = \alpha_m [\alpha_n, x^{\mu} p^{\nu}] + [\alpha_m, x^{\mu} p^{\nu}] \alpha_n$$

$$= \alpha_m \left(x^{\mu} \cancel{[\alpha_n, p^{\nu}]} + [\alpha_n, x^{\mu}] p^{\nu} \right)$$

$$+ \left(x^{\mu} \cancel{[\alpha_m, p^{\nu}]} + [\alpha_m, x^{\mu}] p^{\nu} \right) \alpha_n$$

$$[x^{\mu}, p^{\nu}] = i \eta^{\mu\nu} \rightsquigarrow [x^{\mu}, \alpha_0^{\nu}] = C \cdot i \eta^{\mu\nu}$$

$$= -i \alpha_m^{\mu} \alpha_0^{\nu} \delta_{n,0} - i \alpha_0^{\nu} \alpha_n^{\mu} \delta_{m,0}$$

✓

What about $[p^{\mu}, p^{\nu}]$? vanishes

$$= -\frac{i}{2} \left\{ \alpha_m^{\mu} \alpha_0^{\nu} + \cancel{\alpha_0^{\nu} \alpha_0^{\mu} \delta_{m,0}} + \cancel{\alpha_0^{\mu} \alpha_0^{\nu} \delta_{m,0}} + \alpha_0^{\nu} \alpha_m^{\mu} - \alpha_m^{\nu} \alpha_0^{\mu} - \cancel{\alpha_0^{\mu} \alpha_0^{\nu} \delta_{m,0}} - \cancel{\alpha_0^{\nu} \alpha_0^{\mu} \delta_{m,0}} - \alpha_0^{\mu} \alpha_m^{\nu} \right\}$$

$[\alpha_m^{\mu}, \alpha_0^{\nu}] = 0$

$$= \underline{\underline{-i \left\{ \alpha_m^{\mu} \alpha_0^{\nu} - \alpha_m^{\nu} \alpha_0^{\mu} \right\}}}$$

So in total:

$$[L_m, J^{\mu\nu}] = [L_m, \ell^{\mu\nu} + E^{\mu\nu}] = i \left\{ \alpha_m^{\mu} \alpha_0^{\nu} - \alpha_m^{\nu} \alpha_0^{\mu} \right\} - i \left\{ \alpha_m^{\mu} \alpha_0^{\nu} - \alpha_m^{\nu} \alpha_0^{\mu} \right\} = 0$$

phys. states in the multiplet of Lorentz symmetry

This means that the definition of physical states is invariant under Lorentz