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String Theory Exercise 6 Homework Marvin Zanke

13.11.2018 1) Euclidean d-dimensional ST $\mathbb{R}^{d,0}$, $\eta_{\mu\nu} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$, $\mu, \nu = 1, \dots, d$ $\frac{28}{28}$

Conformal transformation leaves the metric tensor invariant up to a scale, i.e. 1601.03000 Conf. group and 1602.07982 $d \geq 3$ - CFTs

wrong trafo on sheet $\leadsto$ yes

$$\eta'_{\mu\nu}(x'^\mu) = \eta_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = \Lambda(x^\mu)\, \eta_{\mu\nu}(x^\mu) \qquad (*)$$

preserving angles between vectors in ST.

1. Infinitesimal Coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu \pm \epsilon^\mu(x^\mu), \quad \epsilon(x^\mu) \ll 1$$

$$\leadsto \quad \frac{\partial x'^\mu}{\partial x^\nu} = \delta^{\mu\nu} \pm \frac{\partial \epsilon^\mu}{\partial x^\nu} = \delta^{\mu\nu} \pm \partial_\nu \epsilon^\mu$$

and $\frac{\partial x^\nu}{\partial x'^k} = \delta^{\nu k} \mp \partial_k \epsilon^\nu$ s.t.

$$\delta^\mu{}_k \overset{!}{=} \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^k} = (\delta^{\mu\nu} \pm \partial_\nu \epsilon^\mu)(\delta^{\nu k} \mp \partial_k \epsilon^\nu)$$
$$= (\delta^{\mu k} \pm \partial_k \epsilon^\mu \mp \partial_k \epsilon^\mu + \mathcal{O}(\epsilon^2)) \checkmark$$

Why Λ around 1? $\leadsto 1 \in L(x)$, f small because inf. trafo

Requiring that this transformation is conformal with $\Lambda(x^\mu) = (1-f)$ yields (with the upper sign)

$$\eta'_{\mu\nu}(x'^\mu) = \eta_{\rho\sigma}\,(\delta^{\rho\mu} - \partial_\mu \epsilon^\rho)(\delta^{\sigma\nu} - \partial_\nu \epsilon^\sigma) \overset{!}{=} (1-f)\,\eta_{\mu\nu}(x^\mu)$$
$$= \eta_{\mu\nu} - \eta_{\rho\nu} \partial_\mu \epsilon^\rho - \eta_{\mu\sigma} \partial_\nu \epsilon^\sigma + \mathcal{O}(\epsilon^2)$$
$$= \eta_{\mu\nu} - \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu + \mathcal{O}(\epsilon^2)$$
$$\leadsto \quad f\,\eta_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \qquad (**)$$

Taking the trace of this relation, i.e. multiplying $\eta^{\nu\mu}$,

$$f\,\eta^{\nu\mu}\eta_{\mu\nu} = \partial^\nu \epsilon_\nu + \partial^\mu \epsilon_\mu$$
$$\Longleftrightarrow \quad d f = 2(\partial \cdot \epsilon) \quad \leadsto \quad f = \frac{2}{d} \underbrace{(\partial \cdot \epsilon)}_{= \partial_\mu \epsilon^\mu}$$

2. $\partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu = \partial_\rho f \eta_{\mu\nu} = \eta_{\mu\nu} \partial_\rho f$

We also have the permutation of this equation:

$$\eta_{\rho\mu} \partial_\nu f = \partial_\nu \partial_\rho \epsilon_\mu + \partial_\nu \partial_\mu \epsilon_\rho$$
$$\eta_{\nu\rho} \partial_\mu f = \partial_\mu \partial_\nu \epsilon_\rho + \partial_\mu \partial_\rho \epsilon_\nu$$

Combining these:

$$\eta_{\nu\rho} \partial_\mu f + \eta_{\rho\mu} \partial_\nu f - \eta_{\mu\nu} \partial_\rho f = \partial_\mu \partial_\nu \epsilon_\rho + \underline{\partial_\mu \partial_\rho \epsilon_\nu} + \underline{\partial_\nu \partial_\rho \epsilon_\mu} + \partial_\nu \partial_\mu \epsilon_\rho - \underline{\partial_\rho \partial_\mu \epsilon_\nu} - \underline{\partial_\rho \partial_\nu \epsilon_\mu}$$

$$= \partial_\mu \partial_\nu \epsilon_\rho + \partial_\nu \partial_\mu \epsilon_\rho = 2 \partial_\mu \partial_\nu \epsilon_\rho$$

3. Contracting with $\eta^{\mu\nu}$, we find:

$$\eta^{\mu}{}_{\rho} \partial_\mu f + \eta^{\nu}{}_{\rho} \partial_\nu f - d \partial_\rho f = 2 \partial^2 \epsilon_\rho$$
$$\Leftrightarrow \quad \partial_\rho f + \partial_\rho f - d \partial_\rho f = 2 \partial^2 \epsilon_\rho$$
$$\Leftrightarrow \quad (2-d) \partial_\rho f = 2 \partial^2 \epsilon_\rho$$

Taking ∂_ν of this expression, we find:

$$(2-d) \partial_\nu \partial_\rho f = 2 \partial^2 \partial_\nu \epsilon_\rho \rightsquigarrow \partial^2 \partial_\nu \epsilon_\rho = \frac{2-d}{2} \partial_\nu \partial_\rho f \quad (***)$$

We furthermore take ∂^2 of $(**)$ to find

$$\partial^2 \partial_\mu \epsilon_\nu + \partial^2 \partial_\nu \epsilon_\mu = \eta_{\mu\nu} \partial^2 f$$

and then:

$$\eta_{\mu\nu} \partial^2 f = \partial^2 \partial_\mu \epsilon_\nu + \partial^2 \partial_\nu \epsilon_\mu \overset{(***)}{=} \frac{2-d}{2} \partial_\mu \partial_\nu f + \frac{2-d}{2} \partial_\nu \partial_\mu f$$
$$= (2-d) \partial_\mu \partial_\nu f \quad (*^4)$$

$\eta_{\mu\nu}$ flat, i.e. no x-dep.? why then in (1.1) ~~[illegible]~~ is possible that depends on x maybe

Why can exchange part. derivatives?

Contracting once more with $\eta^{\mu\nu}$ yields

$$(2-d)\,\partial^2 f = d\,\partial^2 f \iff 2(1-d)\,\partial^2 f = 0$$
$$\iff (d-1)\,\partial^2 f = 0 \quad (*^5)$$

why (1.7) alone not enough, i.e. what's the problem for $d=2$? ⇝ couldn't even get that eq. if $d=2$.

⇝ no constraint on f for $d=1$.

• $d=2$ considered next exercise

4. For $d \geq 3$, $(*^4)$ and $(*^5)$ imply $\partial_\mu \partial_\nu f = 0$, which means that f can at most be linear in x^μ,

$$f(x^\mu) = A + B_\mu x^\mu .$$

Going back to $(**)$, $f\eta_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$, it is clear that ϵ can then at most be qubic in x^μ, as the derivative of ϵ will constitute to f. We expand

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \quad (*^6)$$

with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$, as $x^\nu x^\rho$ sym. in $\nu \leftrightarrow \rho$. See also:

$$d_{\mu\nu\rho} x^\nu x^\rho = \frac{1}{2}\left(d_{\mu\nu\rho} x^\nu x^\rho + d_{\mu\rho\nu} x^\rho x^\nu\right)$$
$$= \frac{1}{2}\left(d_{\mu\nu\rho} + d_{\mu\rho\nu}\right) x^\nu x^\rho \cancel{=} c_{\mu\nu\rho} x^\nu x^\rho$$

Does a tensor always split in sym. and antisym part? ⇝ should be possible

5. As (1.3) – (1.5) hold for all x^μ, we can treat each power of x^μ in $(*^6)$ seperately.

why can we treat each power seperately? ⇝ if we insert them completely, we will end up w/ an eq. containing powers of x and w/o. x; these can be seperated by powers of x and they have to hold seperately

i) (1.3): $\partial_\mu a_\nu + \partial_\nu a_\mu = f\eta_{\mu\nu} \iff 0 = f\eta_{\mu\nu}$

(1.4): $f = \frac{2}{d}(\partial_\mu a^\mu) = 0$

(1.5): $\underbrace{2\partial_\mu \partial_\nu a_\rho}_{=0} = \eta_{\nu\rho}\partial_\mu f + \eta_{\rho\mu}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f$

⇝ no constraint on a_μ.

ii) (1.3): $\partial_\mu (b_{\nu k} x^k) + \partial_\nu (b_{\mu k} x^k) = f\eta_{\mu\nu}$

$\Longleftrightarrow b_{\nu\mu} + b_{\mu\nu} = f\eta_{\mu\nu}$

(1.4): $f = \frac{2}{d}(\partial_\mu \epsilon^\mu) = \frac{2}{d}\partial_\mu(b^\mu{}_k x^k) = \frac{2}{d} b^\mu_\mu$

(1.5): $\underbrace{2\partial_\mu\partial_\nu(b_{\rho k}x^k)}_{=0} = \eta_{\nu\rho}\partial_\mu f + \eta_{\rho\mu}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f$

For ϵ^μ just pull up the μ index?

$\leadsto$ combining the (1.3) and (1.4) results, we find

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d} b^\rho_\rho \eta_{\mu\nu}$$

iii) (1.3): $\partial_\mu(c_{\nu k\rho}x^k x^\rho) + \partial_\nu(c_{\mu k\rho}x^k x^\rho) = f\eta_{\mu\nu}$

$\Longleftrightarrow c_{\nu\mu\rho}x^\rho + c_{\nu k\mu}x^k + c_{\mu\nu\rho}x^\rho + c_{\mu k\nu}x^k = f\eta_{\mu\nu}$

(1.4): $f = \frac{2}{d}\partial_\mu(c^\mu{}_{\nu k}x^\nu x^k)$

$= \frac{2}{d}(c^\mu{}_{\mu k}x^k + c^\mu{}_{\nu\mu}x^\nu) = \frac{4}{d}c^\mu{}_{\mu k}x^k$

(1.5) $2\partial_\mu\partial_\nu(c_{\rho k\lambda}x^k x^\lambda) = \eta_{\nu\rho}\partial_\mu f + \eta_{\rho\mu}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f$

$\overset{(1.4)}{\Longleftrightarrow} 2\partial_\mu(c_{\rho\nu\lambda}x^\lambda + c_{\rho k\nu}x^k) = \eta_{\nu\rho}(\frac{4}{d}c^\sigma{}_{\sigma\mu}) + \eta_{\rho\mu}(\frac{4}{d}c^\sigma{}_{\sigma\nu})$
$- \eta_{\mu\nu}(\frac{4}{d}c^\sigma{}_{\sigma\rho})$

What about the (1.3) + (1.4) conds. here?

$\Longleftrightarrow 4c_{\rho\mu\nu} = \frac{4}{d}\{\eta_{\nu\rho}c^\sigma{}_{\sigma\mu} + \eta_{\rho\mu}c^\sigma{}_{\sigma\nu} - \eta_{\mu\nu}c^\sigma{}_{\sigma\rho}\}$

$b_\mu = \frac{1}{d}c^\sigma{}_{\sigma\mu}$ $\Longrightarrow c_{\mu\nu\rho} = \eta_{\rho\mu}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu$

$\leadsto$ • a_μ gives rise to an infinitesimal translation

• from (1.9), $b_{\mu\nu}$ splits in antisym. part + trace

$b_{\mu\nu} = m_{\mu\nu} + \alpha\eta_{\mu\nu}$ with $m_{\mu\nu} = -m_{\nu\mu}$

$m_{\mu\nu}$: gives rise to inf. rotations

$\alpha\eta_{\mu\nu}$: gives rise to inf. scale transformation

Why can $b_{\mu\nu}$ (not b_μ?!) be seperated into antisym. part + trace?

• infinitesimal trafo associated to $c_{\mu\nu\rho}$ is given by

$$x'^\mu = x^\mu + 2(b\cdot x)x^\mu - b^\mu x^2 \quad \text{(SCT)}$$

Why trace gives rise to inf. scale trafo etc?

why/how get a finite trafo from inf.?

From the finite transformations belonging to the infinitesimal ones, the generators of the conformal group can be identified as follows (transl. + rotation form the Poincaré group):

Translation: $x'^\mu = x^\mu + a^\mu$ $\rightsquigarrow$ $P_\mu = -i\partial_\mu$

Rotation: $x'^\mu = M^\mu{}_\nu x^\nu$ $\rightsquigarrow$ $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$

Dilation: $x'^\mu = \alpha x^\mu$ $\rightsquigarrow$ $D = -ix^\mu\partial_\mu$

SCT: $x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2x^2}$ $\rightsquigarrow$ $K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$

6.

It follows, using $[x_\mu, P_\nu] = i\eta_{\mu\nu} = -i[x_\mu, \partial_\nu]$

$$[D, P_\mu] = -i[x^k\partial_k, P_\mu] = [x^kP_k, P_\mu]$$
$$= x^k\underbrace{[P_k, P_\mu]}_{=0} + [x_k, P_\mu]P^k = i\eta_{k\mu}P^k = iP_\mu$$

$$[D, K_\mu] = -i[D, 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu]$$
$$= 2[D, x_\mu x^\nu P_\nu] - [D, x^2P_\mu]$$
$$= 2[x^kP_k, x_\mu x^\nu P_\nu] - x^2[D, P_\mu] - [D, x^2]P_\mu$$
$$= 2x_\mu\underbrace{[x^kP_k, x^\nu P_\nu]}_{=0} + 2[x^kP_k, x_\mu]x^\nu P_\nu$$
$$-ix^2P_\mu - [x^kP_k, x^\lambda x_\lambda]P_\mu$$
$$= 2x^k[P_k, x_\mu]x^\nu P_\nu - ix^2P_\mu - x^k[P_k, x^\lambda x_\lambda]P_\mu$$
$$= -2ix_\mu x^\nu P_\nu - ix^2P_\mu - x^kx^\lambda[P_k, x_\lambda]P_\mu - x^k[P_k, x_\lambda]x^\lambda P_\mu$$
$$= -2ix_\mu x^\nu P_\nu - ix^2P_\mu + ix^2P_\mu + ix^2P_\mu$$
$$= -i(2x_\mu x^\nu P_\nu - x^2P_\mu) = (-i)(-i)(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$$
$$= -iK_\mu$$

$$[K_\mu, P_\nu] = [2x_\mu x^kP_k - x^2P_\mu, P_\nu] = -2[P_\nu, x_\mu x^kP_k] + [P_\nu, x^2P_\mu]$$
$$= -2[P_\nu, x_\mu x_k]P^k + [P_\nu, x^2]P_\mu$$
$$= -2x_\mu[P_\nu, x_k]P^k - 2[P_\nu, x_\mu]x_kP^k + x^\lambda[P_\nu, x_\lambda]P_\mu + [P_\nu, x_\lambda]x^\lambda P_\mu$$
$$= 2ix_\mu P_\nu + 2i\eta_{\nu\mu}x_kP^k - ix_\nu P_\mu - ix_\nu P_\mu$$
$$= 2i\eta_{\mu\nu}D + 2i(x_\mu P_\nu - x_\nu P_\mu) = 2i(\eta_{\mu\nu}D - ix_\mu\partial_\nu + ix_\nu\partial_\mu)$$
$$= 2i(\eta_{\mu\nu}D - L_{\mu\nu})$$

$$[K_\rho, L_{\mu\nu}] = [K_\rho, x_\nu P_\mu - x_\mu P_\nu] = [K_\rho, x_\nu P_\mu] - (\mu \leftrightarrow \nu)$$
$$= x_\nu [K_\rho, P_\mu] + [K_\rho, x_\nu] P_\mu - (\mu \leftrightarrow \nu)$$
$$= 2i x_\nu (\eta_{\rho\mu} D - L_{\rho\mu}) + [2 x_\rho x^\kappa P_\kappa - x^2 P_\rho, x_\nu] P_\mu - (\mu \leftrightarrow \nu)$$
$$= 2i x_\nu (\eta_{\rho\mu} D - L_{\rho\mu}) + 2 x_\rho x^\kappa [P_\kappa, x_\nu] P_\mu - x^2 [P_\rho, x_\nu] P_\mu - (\mu \leftrightarrow \nu)$$
$$= 2i x_\nu (\eta_{\rho\mu} D - L_{\rho\mu}) - 2i x_\rho x_\nu P_\mu + i \eta_{\rho\nu} x^2 P_\mu - (\mu \leftrightarrow \nu)$$
$$= 2i x_\nu x^\kappa P_\kappa \eta_{\rho\mu} - 2i x_\nu (x_\mu P_\rho - x_\rho P_\mu) - 2i x_\rho x_\nu P_\mu + i \eta_{\rho\nu} x^2 P_\mu - (\mu \leftrightarrow \nu)$$
$$= 2i x_\nu x^\kappa P_\kappa \eta_{\rho\mu} - 2i x_\nu x_\mu P_\rho + i \eta_{\rho\nu} x^2 P_\mu - (\mu \leftrightarrow \nu)$$
$$= 2 x_\nu x^\kappa \partial_\kappa \eta_{\rho\mu} - 2 x_\nu x_\mu \partial_\rho + \eta_{\rho\nu} x^2 \partial_\mu - \underline{(\mu \leftrightarrow \nu)}$$
$$= 2 x_\nu x^\kappa \partial_\kappa \eta_{\rho\mu} - 2 x_\mu x^\kappa \partial_\kappa \eta_{\rho\nu} + \eta_{\rho\nu} x^2 \partial_\mu - \eta_{\rho\mu} x^2 \partial_\nu$$
$$= i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu)$$

Furthermore, one finds: $[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho})$$

7. Define $J_{\mu\nu} = L_{\mu\nu}$, $J_{-1\mu} = \frac{1}{2}(P_\mu - K_\mu) = -J_{\mu -1}$

$J_{-1,0} = D = -J_{0,-1}$, $J_{0\mu} = \frac{1}{2}(P_\mu + K_\mu) = -J_{\mu 0}$

We claim, that these generators satisfy the $SO(d+1,1)$ algebra

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac})$$

with $\eta_{ab} = \mathrm{diag}(-1, 1, \dots, 1)$

Before the $\eta_{ab} = \mathrm{diag}(1,1,\dots)$?

• "$ab \hat{=} \mu\nu$", "$cd \hat{=} \kappa\lambda$" indices from above

$$[L_{ab}, L_{cd}] = i(\eta_{bc}L_{ad} + \eta_{ad}L_{bc} - \eta_{ac}L_{bd} - \eta_{bd}L_{ac})$$
$$= i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac})$$

What about $\mu,-1$?

• "$ab \hat{=} \mu\nu$", $cd \hat{=} -1\mu$

$$[L_{ab}, \tfrac{1}{2}(P_d - K_d)] = \tfrac{1}{2}([L_{ab}, P_d] - [L_{ab}, K_d])$$
$$= \tfrac{1}{2}i(\eta_{da}K_b - \eta_{db}K_a - \eta_{da}P_b + \eta_{db}P_a)$$
$$= \tfrac{1}{2}i\eta_{db}(P_a - K_a) - \tfrac{1}{2}i\eta_{da}(P_b - K_b)$$
$$= i\eta_{db}J_{-1a} - i\eta_{da}J_{-1b}$$
$$= i(-\eta_{ad}J_{cb} + \underbrace{\eta_{bc}}_{0,\,c=-1}J_{ad} - \underbrace{\eta_{ac}}_{0,\,c=-1}J_{bd} + \eta_{bd}J_{ca})$$
$$= i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ca})$$

• "$ab \hat{=} \mu\nu$", "$cd \hat{=} -10$"

$$[L_{ab}, D] = [x_\nu P_\mu - x_\mu P_\nu, D] = [D, x_\mu P_\nu] - [D, x_\nu P_\mu]$$
$$= ix_\mu P_\nu + [D, x_\mu]P_\nu - ix_\nu P_\mu - [D, x_\nu]P_\mu$$

$[D, x_\mu] = [x^k P_k, x_\mu] = x^k[P_k, x_\mu]$
$= -ix^k\eta_{k\mu} = -ix_\mu$

$$= ix_\mu P_\nu - ix_\mu P_\nu - ix_\nu P_\mu + ix_\nu P_\mu = 0$$
$$= i(\underbrace{\eta_{ad}}_{0,\,\text{off-diag}}J_{bc} + \underbrace{\eta_{bc}}_{0,\,c=-1}J_{ad} - \underbrace{\eta_{ac}}_{0,\,c=-1}J_{bd} - \underbrace{\eta_{bd}}_{0,\,\text{off-diag}}J_{ac})$$

• $ab \hat{=} \mu\nu$, $cd = 0,\mu$

$$[L_{ab}, \tfrac{1}{2}(P_d + K_d)] = \tfrac{1}{2}[L_{ab}, P_d] + \tfrac{1}{2}[L_{ab}, K_d]$$
$$= \tfrac{1}{2}i(\eta_{db}P_a - \eta_{da}P_b + \eta_{db}K_a - \eta_{da}K_b)$$
$$= \tfrac{1}{2}i\eta_{db}(P_a + K_a) - \tfrac{1}{2}i\eta_{da}(P_b + K_b)$$
$$= i\eta_{db}J_{0a} - i\eta_{da}J_{0b}$$
$$= i(-\eta_{ad}J_{cb} + \underbrace{\eta_{bc}}_{0}J_{ad} - \underbrace{\eta_{ac}}_{0}J_{bd} + \eta_{bd}J_{ca})$$
$$= i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac})$$

• $ab \hat{=} -1\mu$, $cd \hat{=} -1\nu$

$$[\tfrac{1}{2}(P_b - K_b), \tfrac{1}{2}(P_d - K_d)] = \tfrac{1}{4}\{[P_b, P_d] - [P_b, K_d] - [K_b, P_d] + [K_b, K_d]\}$$
$$= \tfrac{1}{4}\{2i(\eta_{db}D - L_{db}) - 2i(\eta_{bd}D - L_{bd}) + [K_b, K_d]\}$$

$$[K_b, K_d] = [2x_bx^\kappa P_\kappa - x^2P_b, 2x_dx^\lambda P_\lambda - x^2P_d]$$
$$= 4[x_bx^\kappa P_\kappa, x_dx^\lambda P_\lambda] + [x^2P_b, x^2P_d] - 2[x_bx^\kappa P_\kappa, x^2P_d] - 2[x^2P_b, x_dx^\lambda P_\lambda]$$
$$= 4\{x_b[x^\kappa P_\kappa, x_dx^\lambda P_\lambda] + [x_b, x_dx^\lambda P_\lambda]x^\kappa P_\kappa\}$$
$$+ x^2[P_b, x^2P_d] + [x^2, x^2P_d]P_b$$
$$- 2\{x_b[x^\kappa P_\kappa, x^2P_d] + [x_b, x^2P_d]x^\kappa P_\kappa\}$$
$$- 2\{x_d[x^2P_b, x^\lambda P_\lambda] + [x^2P_b, x_d]x^\lambda P_\lambda\}$$
$$= 4\{x_b[x^\kappa P_\kappa, x_d]x^\lambda P_\lambda + x_d[x_b, x^\lambda P_\lambda]x^\kappa P_\kappa\}$$
$$+ x^2[P_b, x^2]P_d + x^2[x^2, P_d]P_b$$
$$- 2\{-ix_bx^2P_d + ix_dx^2P_b + i\eta_{bd}x^2x^\lambda P_\lambda - i\eta_{bd}x^2x^\lambda P_\lambda\}$$
$$= 4\{-ix_bx_dx^\lambda P_\lambda + ix_dx_bx^\kappa P_\kappa\}$$
$$- 2ix^2x_bP_d + 2ix^2x_dP_b$$
$$+ 2ix_bx^2P_d - 2ix_dx^2P_b = 0$$

$$[x^\kappa P_\kappa, x^2P_d]$$
$$= x^\kappa[P_\kappa, x^2P_d] + [x^\kappa, x^2P_d]P_\kappa$$
$$= x^\kappa[P_\kappa, x^2]P_d + x^2[x^\kappa, P_d]P_\kappa$$
$$= x^\kappa x^\lambda[P_\kappa, x_\lambda]P_d$$
$$+ x^\kappa[P_\kappa, x_\lambda]x^\lambda P_d + ix^2P_d$$
$$= -ix^2P_d$$

and

$$[x_b, x^2P_d] = x^2[x_b, P_d]$$
$$= i\eta_{bd}x^2$$

$$= \tfrac{1}{2}i(L_{bd} - L_{db}) = iL_{bd}$$
$$= i(\underbrace{\eta_{ad}}_{0}J_{bc} + \underbrace{\eta_{bc}}_{0}J_{ad} - \underbrace{\eta_{ac}}_{-1}\underbrace{J_{bd}}_{L_{bd}} - \underbrace{\eta_{bd}}_{0}J_{ac})$$

What J_{-1-1}?

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• "ab ≙ −1μ", "cd = −10"

$$\left[\tfrac{1}{2}(P_b - K_b), D\right] = \tfrac{1}{2}\left\{-iP_b + (-iK_b)\right\} = -\tfrac{1}{2} i (P_b + K_b)$$

$$= -i J_{0b} = i J_{b0}$$

$$= i(\underset{\overset{\shortparallel}{0}}{\eta_{ad}} J_{bc} + \underset{\overset{\shortparallel}{0}}{\eta_{bc}} J_{ad} - \underset{\overset{\shortparallel}{-1}}{\eta_{ac}} \underset{\overset{\shortparallel}{J_{b0}}}{J_{bd}} - \underset{\overset{\shortparallel}{0}}{\eta_{bd}} J_{ac})$$

• "ab ≙ −1μ", "cd ≙ 0μ"

$$\left[\tfrac{1}{2}(P_b - K_b), \tfrac{1}{2}(P_d + K_d)\right] = \tfrac{1}{4}\left\{\underbrace{[P_b, P_d]}_{=0} + [P_b, K_d] - [K_b, P_d] - \underbrace{[K_b, K_d]}_{=0}\right\}$$

$$= -\tfrac{1}{4}\left\{2i(\eta_{db} D - L_{db}) + 2i(\eta_{bd} D - L_{bd})\right\}$$

$$\overset{L_{bd} = -L_{db}}{=} -\tfrac{1}{2} i(\eta_{db} D + \eta_{bd} D) = -i\eta_{bd} D$$

$$= i(\underset{\overset{\shortparallel}{0}}{\eta_{ad}} J_{bc} + \underset{\overset{\shortparallel}{0}}{\eta_{bc}} J_{ad} - \underset{\overset{\shortparallel}{0}}{\eta_{ac}} J_{bd} - \eta_{bd} \underset{\overset{\shortparallel}{J_{-10} = D}}{J_{ac}})$$

D_{-10} w/ nothing else, only $D_{0\mu}$?

• "ab ≙ −10", "cd ≙ 0μ"

$$\left[D, \tfrac{1}{2}(P_d + K_d)\right] = \tfrac{1}{2}\left\{[D, P_d] + [D, K_d]\right\}$$

$$= \tfrac{1}{2}\left\{iP_d - iK_d\right\} = i J_{-1d}$$

$$= i(\underset{\overset{\shortparallel}{0}}{\eta_{ad}} J_{bc} + \underset{\overset{\shortparallel}{+1}}{\eta_{bc}} \underset{\overset{\shortparallel}{J_{-1d}}}{J_{ad}} - \underset{\overset{\shortparallel}{0}}{\eta_{ac}} J_{bd} - \underset{\overset{\shortparallel}{0}}{\eta_{bd}} J_{ac})$$

$\eta_{00} = 1$, not -1?

• "ab ≙ 0μ", "cd ≙ 0ν"

$$\left[\tfrac{1}{2}(P_b + K_b), \tfrac{1}{2}(P_d + K_d)\right] = \tfrac{1}{4}\left\{\underbrace{[P_b, P_d]}_{=0} + [P_b, K_d] + [K_b, P_d] + \underbrace{[K_b, K_d]}_{=0}\right\}$$

$$= \tfrac{1}{4}\left\{-2i(\eta_{db} D - L_{db}) + 2i(\eta_{bd} D - L_{bd})\right\}$$

$$= i L_{db} = -i L_{bd}$$

$$= i(\underset{\overset{\shortparallel}{0}}{\eta_{ad}} J_{bc} + \underset{\overset{\shortparallel}{0}}{\eta_{bc}} J_{ad} - \underset{\overset{\shortparallel}{+1}}{\eta_{ac}} \underset{\overset{\shortparallel}{L_{bd}}}{J_{bd}} - \underset{\overset{\shortparallel}{0\text{, as bcd vanishes trivially}}}{\eta_{bd}} J_{ac})$$

2)

1. $\delta_{\epsilon(z_1),\epsilon(z_2)} G(z_1,z_2) = \delta_{\epsilon(z_1)\epsilon(z_2)} \langle \phi_1(z_1)\, \phi_2(z_2) \rangle$

$= \langle \delta_{\epsilon(z_1)} \phi_1(z_1)\, \phi_2(z_2) \rangle + \langle \phi_1(z_1)\, \delta_{\epsilon(z_2)} \phi_2(z_2) \rangle$

$= \langle -(\epsilon(z_1)\partial_{z_1} + h_1 \partial_{z_1}\epsilon(z_1))\phi_1(z_1)\, \phi_2(z_2) \rangle$
$+ \langle \phi_1(z_1)\, (-1)(\epsilon(z_2)\partial_{z_2} + h_2 \partial_{z_2}\epsilon(z_2))\phi_2(z_2) \rangle$

Question of def. how it transforms?

$= -[\epsilon(z_1)\partial_{z_1} + h_1 \partial_{z_1}\epsilon(z_1)]\, G(z_1,z_2)$
$-[\epsilon(z_2)\partial_{z_2} + h_2 \partial_{z_2}\epsilon(z_2)]\, G(z_1,z_2)$

Global conf. trafo $\to$ derivative vanishes?

$= -[\epsilon(z_1)\partial_{z_1} + h_1 \partial_{z_1}\epsilon(z_1) + \epsilon(z_2)\partial_{z_2} + h_2 \partial_{z_2}\epsilon(z_2)]\, G(z_1,z_2)$

$\overset{!}{=} 0$

Why only Lorentz group $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq SO(3,1)$?

2. $\epsilon(z_i) = \alpha \rightsquigarrow (\alpha \partial_{z_1} + \alpha \partial_{z_2}) G(z_1,z_2) \overset{!}{=} 0$

$\iff (\partial_{z_1} + \partial_{z_2}) G(z_1,z_2) \overset{!}{=} 0$

Also holomorph if it only depends on $\bar{z}$?

Introduce a new basis $z_+ = z_1 + z_2$, $z_- = z_1 - z_2$ $\rightsquigarrow$ $z_1 = \frac{1}{2}(z_+ + z_-)$, $z_2 = \frac{1}{2}(z_+ - z_-)$

Why $\bar{\epsilon}(\bar{z})\partial_{\bar{z}}$?

$\rightsquigarrow \partial_{z_1} = \frac{\partial}{\partial z_1} = \frac{\partial}{\partial z_+}\frac{\partial z_+}{\partial z_1} + \frac{\partial}{\partial z_-}\frac{\partial z_-}{\partial z_1} = \partial_{z_+} + \partial_{z_-}$

$\partial_{z_2} = \frac{\partial}{\partial z_2} = \frac{\partial}{\partial z_+}\frac{\partial z_+}{\partial z_2} + \frac{\partial}{\partial z_-}\frac{\partial z_-}{\partial z_2} = \partial_{z_+} - \partial_{z_-}$

$\rightsquigarrow 2\partial_{z_+} G(z_1,z_2) \overset{!}{=} 0 \rightsquigarrow$ No dependence on z_+

$\rightsquigarrow$ at most depends on z_-.

3. $\epsilon(z_i) = \beta z_i \rightsquigarrow [\beta z_1 \partial_{z_1} + h_1\beta + \beta z_2 \partial_{z_2} + h_2 \beta]\, G(z_1 - z_2) \overset{!}{=} 0$

$\rightsquigarrow [z_1(\partial_{z_+} + \partial_{z_-}) + h_1 + z_2(\partial_{z_+} - \partial_{z_-}) + h_2]\, G(z_-) \overset{!}{=} 0$

$\rightsquigarrow [z_-\partial_{z_-} + h_1 + h_2]\, G(z_-) \overset{!}{=} 0$

$\rightsquigarrow z_- \partial_{z_-} G(z_-) = -(h_1+h_2) G(z_-)$

$\iff \frac{G'(z_-)}{G(z_-)} = -\frac{h_1+h_2}{z_-}$

$\cancel{\rightsquigarrow \ln G(z_-) = -(h_1+h_2)\ln z_- + C}$

$$\rightsquigarrow \int_{z_0}^{z_1-z_2} \frac{G'(z_-)}{G(z_-)} dz_- = -\int_{z_0}^{z_1-z_2} \frac{h_1+h_2}{z_-} dz_-$$

$$\rightsquigarrow \ln G(z_1-z_2) - \ln G(z_0) = -(h_1+h_2)\{\ln(z_1-z_2) - \ln(z_0)\}$$

$$\Longleftrightarrow G(z_1-z_2) = e^{-(h_1+h_2)\ln(z_1-z_2)} \cdot \underbrace{e^{\ln G(z_0) + (h_1+h_2)\ln(z_0)}}_{C_{12} = G(z_0)\cdot z_0^{h_1+h_2}}$$

$$\Longleftrightarrow G(z_1-z_2) = \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}}$$

4. $\varepsilon(z_i) = \gamma z_i^2 \rightsquigarrow [\gamma z_1^2 \partial_{z_1} + 2h_1\gamma z_1 + \gamma z_2^2 \partial_{z_2} + 2h_2 \gamma z_2] G(z_1-z_2) \stackrel{!}{=} 0$

insert $G(z_1-z_2)$, divide by C_{12}

$$\rightsquigarrow -z_1^2(h_1+h_2)(z_1-z_2)^{-(h_1+h_2)-1} + 2h_1 z_1 (z_1-z_2)^{-(h_1+h_2)}$$
$$+ z_2^2 (h_1+h_2)(z_1-z_2)^{-(h_1+h_2)-1} + 2h_2 z_2 (z_1-z_2)^{-(h_1+h_2)}$$

$$\rightsquigarrow \frac{h_1+h_2}{(z_1-z_2)^{h_1+h_2+1}}\{z_2^2 - z_1^2\} + (2h_1 z_1 + 2h_2 z_2)(z_1-z_2)^{-(h_1+h_2)}$$

$$\Longleftrightarrow \frac{-(h_1+h_2)(z_1+z_2)}{(z_1-z_2)^{h_1+h_2}} + \frac{2h_1 z_1 + 2h_2 z_2}{(z_1-z_2)^{h_1+h_2}} \stackrel{!}{=} 0$$

$$\Longleftrightarrow \frac{-h_1 z_1 - h_1 z_2 - h_2 z_1 - h_2 z_2 + 2h_1 z_1 + 2h_2 z_2}{(z_1-z_2)^{h_1+h_2}} \stackrel{!}{=} 0$$

$$\Longleftrightarrow \frac{h_1 z_1 - h_1 z_2 + h_2 z_2 - h_2 z_1}{(z_1-z_2)^{h_1+h_2}} \stackrel{!}{=} 0$$

$$\Longleftrightarrow \{z_1(h_1-h_2) + z_2(h_2-h_1)\} \frac{G(z_1-z_2)}{C_{12}} \stackrel{!}{=} 0$$

$$\rightsquigarrow h_1 = h_2 \quad \text{or} \quad G(z_1-z_2) = 0$$

$$\rightsquigarrow \langle \phi_1(z_1)\phi_2(z_2)\rangle = \begin{cases} \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}} & h_1 = h_2 \\ 0 & \text{else} \end{cases}$$