Disclaimer

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I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

String Theory Exercise 7 Homework

Marvin Zanke

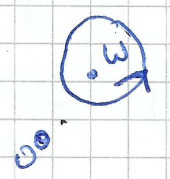
10 + 8 + 7 + 18 = 40

1) 1. Having to general operators $A(z)$, $B(z)$, we define

$$A = \oint_{C_0} \frac{dz}{2\pi i} A(z) \quad , \quad B = \oint_{C_0} \frac{dz}{2\pi i} B(z)$$

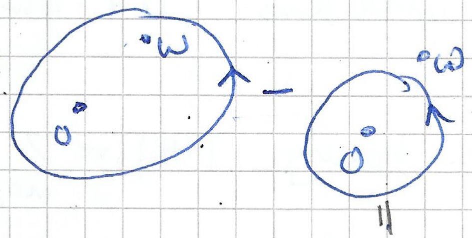
For some fixed w, we consider the integration of a radial ordered expression on a contour around w, going counter-clockwise:

$$\oint_{C_w} \frac{dz}{2\pi i} R(A(z) B(w)) \quad \hat{=}$$



Does the R belong outside or inside of the integral → inside; always of a product, tells you how to close a contour.

$$= \oint_{C_1} \frac{dz}{2\pi i} A(z) B(w) - \oint_{C_2} \frac{dz}{2\pi i} B(w) A(z) \quad \hat{=}$$

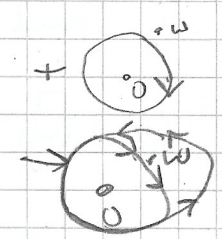


limit of C_1 and C_2 being close to w, i.e. $\epsilon \to 0$

Last step → why contour now C_0? → two contours, one $w+\epsilon$, one $w-\epsilon$, take limit $\epsilon \to 0$ afterwards, so it's no problem that contour goes through w.

$$= \oint_{C_0} \frac{dz}{2\pi i} [A(z), B(w)]$$

$$= [A, B(w)] \quad (*)$$



Integrating w/ respect to w once more on $(*)$, we find

$$\oint_{C_0} \frac{dw}{2\pi i} [A, B(w)] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} R(A(z) B(w))$$

$= [A, B]$ we find the ~~relation~~ on the sheet.

2. $$\delta_\epsilon \phi(w) = -[Q_\epsilon, \phi(w)] = -\oint_{C_0} \frac{dz}{2\pi i} [\epsilon(z) T(z), \phi(w)]$$

$$Q_\epsilon = \oint_{C_0} \frac{dz}{2\pi i} \epsilon(z) T(z)$$

$$= -\oint_{C_w} \frac{dz}{2\pi i} R(\epsilon(z) T(z) \phi(w))$$

Also, $\delta_\epsilon \phi(w) \overset{(-)}{=} (h\, \partial_w \epsilon(w) + \epsilon(w) \partial_w) \phi(w)$ from the lecture

Inserting the Cauchy-Riemann formula for ϵ and $\partial\epsilon$, we find:

$$-\delta_\epsilon \phi(w) = \left\{ h \oint_{C_w} \frac{dz}{2\pi i} \frac{\epsilon(z)}{(z-w)^2} + \oint_{C_w} \frac{dz}{2\pi i} \frac{\epsilon(z)}{(z-w)} \partial_w \right\} \phi(w)$$

regular terms should already be here?

$$= \oint_{C_w} \frac{dz}{2\pi i} \left\{ h \frac{\epsilon(z)\phi(w)}{(z-w)^2} + \frac{\epsilon(z)\partial_w \phi(w)}{z-w} + \text{reg} \right\}$$

reg. terms give zero on integration

In total, we thus get

$$-R(\epsilon(z) T(z)\phi(w)) = -\epsilon(z) \left\{ \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{(z-w)} \right\} + \text{reg}$$

✓ Okay to turn path of int. around? Otherwise wrong sign?

and thus $\quad T(z)\phi(w) = \frac{h\phi(w)}{\cancel{(z-w)^2}} + \frac{\partial_w \phi(w)}{z-w} + \text{reg}$

~~..~~ not needed, (-) sign in δ_ϵ as well

3. Having $\quad T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$

we find

Can everything be written as a Laurent series?

$$[L_m, L_n] = \left[\oint_{C_0} \frac{dz}{2\pi i} z^{m+1} T(z), \oint_{C_0} \frac{dw}{2\pi i} w^{n+1} T(w) \right]$$

$$= \oint \frac{dz}{2\pi i} \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)]$$

$$= \oint_{C_w} \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} R(T(z)T(w))$$

the expression is radial ordered on sheet

✓ How to get $T(z)T(w)$? And it's radial ordered on sheet?

$$= \oint \frac{dz}{2\pi i} \frac{dw}{2\pi i} z^{m+1} w^{n+1} \left\{ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{fin} \right\}$$

~~..~~ To get it, use Wick's theorem. If nothing is written there radial ordered is meant

the finite terms vanish due to the contour int.

$$\overset{\downarrow}{=} \oint \frac{dw}{2\pi i} w^{n+1} \oint_{C_w} \frac{dz}{2\pi i} \left\{ \frac{c/2 \, z^{m+1}}{(z-w)^4} + \frac{2T(w) z^{m+1}}{(z-w)^2} + \frac{z^{m+1} \partial_w T(w)}{z-w} \right\}$$

C.R.

$$\overset{\downarrow}{=} \oint \frac{dw}{2\pi i} w^{n+1} \left\{ \frac{c}{12} (m+1)m(m-1) w^{m-2} + 2(m+1) T(w) w^m + \partial_w T(w) w^{m+1} \right\}$$

$$= \oint \frac{dw}{2\pi i} \left\{ \frac{c}{12} (m+1)m(m-1) w^{m+n-1} + 2(m+1) T(w) w^{m+n+1} + w^{m+n+2} \partial_w T(w) \right\}$$

$$- \oint \frac{dw}{2\pi i} \left\{ \frac{c}{12} m(m^2-1) w^{m+n-1} + 2(m+1) T(w) w^{m+n+1} + w^{m+n+2} \partial_w T(w) \right\}$$

$$= \oint \frac{dw}{2\pi i} \left\{ \frac{c}{12} m(m^2-1) w^{m+n-1} + 2(m+1) \sum_k w^{m+n-k-1} L_k - w^{m+n+2} \sum_k (k+2) w^{-k-3} L_k \right\}$$

C.R. for $m+n \geq 0$ and $m+n-k \geq 0$ for $m+n \geq 0$ and $m+n-k \geq 0$ as well as $m+n < 0$, $m+n-k < 0$ respectively, the integrand is either regular or has a vanishing residue and yields zero

$$= \frac{c}{12} m(m^2-1) \delta_{m+n,0} + \{ 2(m+1) - (m+n+2) \} L_{m+n}$$

$$= \frac{c}{12} m(m^2-1) \delta_{m+n,0} + (m-n) \cancel{L_{m+n}}$$

That doesn't mean that $T(z)T(w)$ is eq. to the Virasoro alg.?

2) Having $S = \frac{g}{2} \int d^2x \left(\partial_\alpha \varphi \partial^\alpha \varphi + m^2 \varphi^2 \right)$,

we are looking for the 2-point fct., i.e. the propagator $K(x,y) = \langle \varphi(x) \varphi(y) \rangle$.

It obeys $$g\left(-\partial_x^2 + m^2\right) K(x,y) = \delta(x-y)$$

Also show that ϕ transforms as primary field → use prior exercise? ⇝ ϕ is not a primary field $T\phi \neq \frac{h\phi}{(z-w)^2} + \frac{\partial\phi(w)}{z-w}$

but $\partial\phi$ is: $T\partial\phi = \frac{1}{(z-w)^2} + \frac{\partial(\partial\phi)}{z-w}$

$\langle \phi\phi \rangle \sim \log|x-y|$, ✓

$\langle \partial\phi\partial\phi \rangle \sim \frac{1}{(x-y)^2}$ ✓

Okay to choose $y=0$? ⇝ also in tutorial (y as ref. point)

✓ 1. Because of translational variance, we have
$K(x,y) = K(x-y)$ and because of rotational inv.
$K(x,y) = K(|x-y|) = K(r)$, $r := |x-y|$

From now on, $m=0$. We show that the solution is then given by $K(r) = -\frac{1}{2\pi g}\log r + \text{const}$

$$-g\,\partial_x^2 K(x,y) = \delta(x-y)$$

We change to spherical coordinates, where
$$\partial_x^2 = \partial_{x_1}^2 + \partial_{x_2}^2 = \frac{1}{r}\partial_r(r\partial_r) + \underbrace{\frac{1}{r^2}\partial_\varphi^2}_{=0 \text{ in our case, only dependent on difference}}$$

$$\rightsquigarrow -g\int_{D_R} d^2A \left(\frac{1}{r}\partial_r\left(r\partial_r K(r)\right)\right) = \int_{D_R} d^2A \underbrace{\delta(r)\delta(\varphi)\frac{1}{r}}_{\frac{1}{|\det J|}\delta(r)\delta(\varphi) \text{ in sph. coord.}}$$

as $\int d^2x\, \delta^{(2)}(x) \overset{!}{=} 1 \rightsquigarrow \int dr\, d\varphi\, r\, \delta^{(2)}(x) \overset{!}{=} 1$
$\rightsquigarrow \delta^2(x) = \frac{1}{r}\delta(\varphi)\delta(r)$

$$\rightsquigarrow -g\int_0^R dr \int_0^{2\pi} d\varphi\, \partial_r\left(r\partial_r K(r)\right) = \int_0^R dr \int_0^{2\pi} d\varphi\, \delta(r)\delta(\varphi) = 1$$

$$\rightsquigarrow -2\pi g\left[r\partial_r K(r)\right]\Big|_0^R \overset{!}{=} 1$$

$\partial_r K(r)$ not regular at zero?

$$\rightsquigarrow R K'(R) - 0\cdot K'(0) = -\frac{1}{2\pi g}$$

$$\rightsquigarrow K'(R) = -\frac{1}{2\pi g R} \rightsquigarrow K(r) = -\frac{1}{2\pi g}\log r \cancel{+C}$$

No. See Tong's lecture about no Goldstone boson in 2d

2. $T(z) = -2\pi g :\partial\varphi\partial\varphi:$, $V_\alpha(z,\bar z) = :e^{2\pi i\alpha\varphi(z,\bar z)}:$

How to get the OPE in compl. coord.?

$$T(z)V_\alpha(w,\bar w) = -2\pi g :\partial\varphi(z)\partial\varphi(z): :e^{2\pi i\alpha\varphi(w,\bar w)}:$$

$$= -2\pi g :\partial\varphi(z)\partial\varphi(z): :\sum_{n=0}^{\infty}\frac{(2\pi i\alpha)^n}{n!} :\varphi^n(w,\bar w):$$

$$= -2\pi g :\partial\varphi(z)\partial\varphi(z): :\sum_{n=0}^{\infty}\frac{(2\pi i\alpha)^n}{n!} :\varphi(w,\bar w)\ldots\varphi(w,\bar w):$$

Why $T(z)$ given by this? → derive from action and demand Wa (see lecture)

$$= -2\pi g \Big\{ N.O. + \sum_{n=0}^{\infty} \underset{\substack{\uparrow\\ \text{choose one } \partial\varphi(z)}}{2}\,\underset{\substack{\uparrow\\ n \text{ possibilities to contract w/ one } \partial\varphi(z)}}{n}\frac{(2\pi i\alpha)^n}{n!} \langle\partial\varphi(z)\varphi(w,\bar w)\rangle :\partial\varphi(z)\varphi^{n-1}(w,\bar w):$$

$$+ \sum_{n=0}^{\infty} \underset{\substack{\uparrow\\ n(n-1) \text{ poss. to contract both } \partial\varphi\text{'s.}}}{n(n-1)}\frac{(2\pi i\alpha)^n}{n!}\langle\partial\varphi(z)\varphi(w,\bar w)\rangle\langle\partial\varphi(z)\varphi(w,\bar w)\rangle \times :\varphi^{n-2}(w,\bar w): \Big\}$$

$$= -2\pi g\Big\{N.O. + 2\pi i\alpha \sum_{n=1}^{\infty}\frac{(2\pi i\alpha)^{n-1}}{(n-1)!}\langle\partial\varphi(z)\varphi(w,\bar w)\rangle :\partial\varphi(z)\varphi^{n-1}(w,\bar w):$$

$$+(2\pi i\alpha)^2\sum_{n=2}^{\infty}\frac{(2\pi i\alpha)^{n-2}}{(n-2)!}\langle\partial\varphi(z)\varphi(w,\bar w)\rangle^2 :\varphi^{n-2}(w,\bar w):$$

$$= -2\pi g\Big\{N.O. + 2\pi i\alpha\langle\partial\varphi(z)\varphi(w,\bar w)\rangle\sum_{n=0}^{\infty}\frac{(2\pi i\alpha)^n}{n!}:\partial\varphi(z)\varphi^n(w,\bar w):$$

$$-4\pi^2\alpha^2\langle\partial\varphi(z)\varphi(w,\bar w)\rangle^2\sum_{n=0}^{\infty}\frac{(2\pi i\alpha)^n}{n!}:\varphi^n(w,\bar w):\Big\}$$

$$= -2\pi g\Big\{N.O. + 2\pi i\alpha\left(-\frac{1}{4\pi g(z-w)}\right)\sum_{n=0}^{\infty}\frac{(2\pi i\alpha)^n}{n!}:\partial\varphi(w)\varphi^n(w,\bar w):$$

$$-4\pi^2\alpha^2\left(\frac{1}{16\pi^2 g^2(z-w)^2}\right):V_\alpha(w,\bar w):\Big\}$$

deriving the normal ordered product w.r.t. w, the factor $2\pi i\alpha$ brings $\partial\varphi$ in the normal ordering

$$= -2\pi g\Big\{N.O. - \frac{1}{4\pi g(z-w)} :\partial V_\alpha(w,\bar w):$$

$$-\frac{\alpha^2}{4g^2(z-w)^2}V_\alpha(w,\bar w)\Big\}$$

$$= N.O. + \frac{1}{2}\frac{1}{z-w}\partial V_\alpha(w,\bar w) + \frac{\alpha^2\pi}{2g}\frac{1}{(z-w)^2}V_\alpha(w,\bar w)$$

Why leave out the NO term in the end? → these are regular terms

$$\Rightarrow h = \bar h = \frac{\alpha^2\pi}{2g}$$

3) $S = \frac{1}{2} g \int d^2x\, \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi$, $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

1. Looking at $\gamma^0 \gamma^\mu \partial_\mu = \gamma^0 (\gamma^0 \partial_0 + \gamma^1 \partial_1)$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_0 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \partial_1 \right\}$$

$$= \partial_0 \mathbb{1} + \partial_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix}$$

we find $S = \frac{1}{2} g \int d^2x\, \Psi^\dagger \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} \Psi$

$\Psi = \begin{pmatrix} \psi \\ \bar\psi \end{pmatrix} \Rightarrow \frac{1}{2} g \int d^2x \left\{ \begin{pmatrix} \psi \\ \bar\psi \end{pmatrix}^\dagger \begin{pmatrix} (\partial_0 + i\partial_1)\psi \\ (\partial_0 - i\partial_1)\bar\psi \end{pmatrix} \right\}$

$$= \frac{1}{2} g \int d^2x \left\{ \psi^* (\partial_0 + i\partial_1)\psi + \bar\psi^* (\partial_0 - i\partial_1)\bar\psi \right\}$$

Defining $z = x + iy$, $\bar z = x - iy \Rightarrow x = \frac{1}{2}(z + \bar z)$, $y = \frac{1}{2i}(z - \bar z)$

$$\Rightarrow \partial_z = \frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}(\partial_0 - i\partial_1)$$

$$\partial_{\bar z} = \frac{\partial}{\partial \bar z} = \frac{\partial x}{\partial \bar z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar z}\frac{\partial}{\partial y} = \frac{1}{2}(\partial_0 + i\partial_1)$$

$$\Rightarrow \partial_0 = \partial_z + \partial_{\bar z} \;,\; \partial_1 = i(\partial_z - \partial_{\bar z})$$

Okay if int measure still dx? $\Rightarrow$ look tutorial, additional factor w/ dz

we find $S = g \int d^2x \left\{ \psi^* \partial_z \psi + \bar\psi^* \partial_{\bar z} \bar\psi \right\}$

Maj. fermion $\psi = \psi^*$ $\rightarrow = g \int d^2x \left\{ \psi \partial_z \psi + \bar\psi \partial_{\bar z} \bar\psi \right\}$

in 2d it means that it is real.

Why the F*! would the eq of motion for ψ be $0=0$?! $\Rightarrow$ see tutorial; Calculate $\delta_\psi S$ and use that ψ is a Grassmann var., anticommutes!

The eq. of motion yields $0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi}$

$$= \partial_z \psi - \partial_z \psi = 0$$

and does not bring us any further. Yes, if we treat ψ and ψ^* as independent, we find

$\partial_z \psi^* = 0$ can not just be compl. conj.?

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = \partial_z \psi^* \Leftrightarrow \partial_{\bar z} \psi = 0$$

Analogously $\partial_z \bar\psi = 0 \Rightarrow \psi$ holomorphic and $\bar\psi$ anti-holomorphic.

If, instead, we worked with Ψ, we find

$$S = \tfrac{1}{2} g \int d^2x\, \Psi^+ \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} \Psi = g \int d^2x\, \Psi^+ \begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} \Psi$$

$$\Rightarrow 0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^+)} - \frac{\partial \mathcal{L}}{\partial \Psi^+} = -g \begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} \Psi$$

$$\Rightarrow \partial_{\bar z} \Psi = 0 \quad \text{and} \quad \partial_z \bar\Psi = 0, \text{ i.e. the same.}$$

Other eq of motion not possible?

2. We want to calculate the correlator $\langle \Psi_i(x), \Psi_j(y) \rangle$, which is the propagator $G_{ij}(x,y)$.

Rewriting $S = \frac{1}{2} \int d^2x\, d^2y\, \Psi_i(x) A_{ij}(x,y) \Psi_j(y)$

What is anti-hol.?

with $A_{ij}(x,y) = g\,\delta(x-y) (\gamma^0\gamma^\mu)_{ij} \partial_\mu$ the kernel

$$\left(\Rightarrow S = \frac{g}{2} \int d^2x\, \Psi^+(x) (\gamma^0\gamma^\mu) \partial_\mu \Psi(x) \;\checkmark \right)$$

We are then looking for the solution of

$$\{ g (\gamma^0\gamma^\mu)_{ij} \partial_\mu \} G_{jk}(x,y) = \delta(x-y)\, \delta_{ik}$$

Why does $\delta(x-y)$ cancel out in A_{ij}?

3. The δ-distribution can be written as follows:

$$\delta(x) = \frac{1}{\pi} \partial_{\bar z} \frac{1}{z} = \frac{1}{\pi} \partial_z \frac{1}{\bar z} \quad \text{(see di Francesco, CFT)}$$

The above eq. in matrix form is then

$$2g \begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} \begin{pmatrix} G_{11}(\substack{z \\ \bar z}, \substack{w \\ \bar w}) & G_{21}(\substack{z \\ \bar z}, \substack{w \\ \bar w}) \\ G_{12}(\substack{z \\ \bar z}, \substack{w \\ \bar w}) & G_{22}(\substack{z \\ \bar z}, \substack{w \\ \bar w}) \end{pmatrix} = \frac{1}{\pi} \begin{pmatrix} \partial_{\bar z} \frac{1}{z-w} & 0 \\ 0 & \partial_z \frac{1}{\bar z - \bar w} \end{pmatrix}$$

where we replaced $x \to (z, \bar z)$, $y \to (w, \bar w)$

Why now $z, \bar z, w, \bar w$? 4 vars before?

$$\Rightarrow G_{11}(\substack{z \\ \bar z}, \substack{w \\ \bar w}) = \frac{1}{2\pi g} \frac{1}{z-w}, \quad G_{22}(\substack{z \\ \bar z}, \substack{w \\ \bar w}) = \frac{1}{2\pi g} \frac{1}{\bar z - \bar w}$$

$$G_{12} = G_{21} = 0$$

4) $S = \frac{g}{2}\int d^2x\, b_{\mu\nu}\partial^\mu c^\nu$ ghost system with

$b_{\mu\nu}$ traceless sym. tensor and $b_{\mu\nu}, c^\mu$ anticommuting fields.

Why traceless sym.?

The propagator can be calculated to be

$b_{\mu\nu}$ and c^μ anticom. under each other?

$$\langle b(z)c(w)\rangle = \frac{1}{\pi g}\frac{1}{z-w}$$

1. $$\langle b(z)\partial c(w)\rangle = \frac{\partial}{\partial w}\langle b(z)c(w)\rangle = \frac{1}{\pi g}\frac{1}{(z-w)^2}$$

$$\langle \partial b(z)c(w)\rangle = \frac{\partial}{\partial z}\langle b(z)c(w)\rangle = -\frac{1}{\pi g}\frac{1}{(z-w)^2}$$

$$\langle \partial b(z)\partial c(w)\rangle = \frac{\partial}{\partial z}\frac{\partial}{\partial w}\cancel{\langle b(z)c(w)\rangle} = -\frac{2}{\pi g}\frac{1}{(z-w)^3}$$

hol. energy-mom. tensor solely means that it only depends on z?

2. $$T(z)b(w) = \pi g :(2\partial c(z)b(z) + c\partial b(z)):\,:b(w):$$

$$= \pi g\,\{\text{N.O.} - 2\langle\partial c(z)b(w)\rangle :b(z): + 2\langle b(z)b(w)\rangle :\partial c(z): - \langle c(z)b(w)\rangle :\partial b(z): + \langle\partial b(z)b(w)\rangle :c(z):\}$$

why $\langle bb\rangle = 0$? And where do reg. terms come in?

$\langle bb\rangle$'s $= 0$ as well as the derivatives of course

$$= \pi g\,\{\text{N.O.} + 2\frac{1}{\pi g}\frac{1}{(z-w)^2}:b(z): - \frac{1}{\pi g}\frac{1}{z-w}:\partial b(z):\}$$

Why does N.O. not contribute to OPE?

up to reg. terms

$$= \frac{2:b(z):}{(z-w)^2} - \frac{:\partial b(z):}{z-w} + \text{reg.}$$

taylor around $z=w$

$$= \frac{2:b(w):}{(z-w)^2} + \frac{2:\partial b(w):(z-w)}{(z-w)^2} - \frac{:\partial b(w):}{z-w} + \text{reg.}$$

$$= \frac{2:b(w):}{(z-w)^2} + \frac{:\partial b(w):}{z-w} \leadsto \cancel{\Delta = 2}$$

Still $:b:$ and $:\partial b:$ in the end?

$$T(z)c(w) = \pi g :2\partial c(z)b(z) + c(z)\partial b(z):\,:c(w):$$

$$= \pi g\,\{\text{N.O.} + 2\langle b(z)b(w)\rangle :\partial c(z): - 2\langle\partial c(z)c(w)\rangle b(z) + \langle\partial b(z)c(w)\rangle :c(z): - \langle c(z)c(w)\rangle :\partial b(z):\}$$

$$= \pi g \left\{ N.O. + 2 \frac{1}{\pi g} \frac{:\partial c(z):}{z-w} - \frac{1}{\pi g} \frac{c(z)}{(z-w)^2} \right\}$$

up to reg. terms

$$\sim \frac{2:\partial c(z):}{z-w} - \frac{c(z)}{(z-w)^2}$$

taylor

$$\cong \frac{2\partial c(w):}{z-w} - \frac{:c(w):}{(z-w)^2} - \frac{:\partial c(w):}{z-w} =$$

$$= - \frac{:c(w):}{(z-w)^2} - \frac{:\partial c(w):}{z-w} \rightsquigarrow \cancel{\Delta =} -1$$

3. $T(z)\,T(w) = \pi^2 g^2 :2\partial c(z) b(z) + c(z)\partial b(z): \; :2\partial c(w) b(w) + c(w)\partial b(w):$

$$= \pi^2 g^2 \{ :2\partial c(z) b(z): :2\partial c(w) b(w): + :2\partial c(z) b(z): :c(w)\partial b(w):$$
$$+ :c(z)\partial b(z): :2\partial c(w) b(w): + :c(z)\partial b(z): :c(w)\partial b(w): \}$$

$$= \pi^2 g^2 \{ 4(\langle \partial c(z) b(w)\rangle \langle b(z)\partial c(w)\rangle + \langle \partial c(z) b(w)\rangle :b(z)\partial c(w):$$
$$+ \langle b(z)\partial c(w)\rangle :\partial c(z) b(w):)$$
$$+2(\langle \partial c(z)\partial b(w)\rangle \langle b(z) c(w)\rangle + \langle \partial c(z)\partial b(w)\rangle :b(z) c(w):$$
$$+ \langle b(z) c(w)\rangle :\partial c(z)\partial b(w):)$$
$$+2(\langle c(z) b(w)\rangle \langle \partial b(z)\partial c(w)\rangle + \langle c(z) b(w)\rangle :\partial b(z)\partial c(w):$$
$$+ \langle \partial b(z)\partial c(w)\rangle : c(z) b(w):)$$
$$+ (\langle c(z)\partial b(w)\rangle \langle \partial b(z) c(w)\rangle + \langle c(z)\partial b(w)\rangle :\partial b(z) c(w):$$
$$+ \langle \partial b(z) c(w)\rangle : c(z)\partial b(w):) \}$$

$$= \pi^2 g^2 \{ 4 \Big[\Big(-\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big)\Big(\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big) + \Big(-\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big) :b(z)\partial c(w):$$
$$+ \Big(\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big) :\partial c(z) b(w): \Big]$$
$$+2 \Big[\Big(-\frac{2}{\pi g}\frac{1}{(z-w)^3}\Big)\Big(\frac{1}{\pi g}\frac{1}{z-w}\Big) + \Big(-\frac{2}{\pi g}\frac{1}{(z-w)^3}\Big) :b(z) c(w):$$
$$+ \Big(\frac{1}{\pi g}\frac{1}{(z-w)}\Big) :\partial c(z)\partial b(w): \Big]$$
$$+2 \Big[\Big(\frac{1}{\pi g}\frac{1}{(z-w)}\Big)\Big(-\frac{2}{\pi g}\frac{1}{(z-w)^3}\Big) + \Big(\frac{1}{\pi g}\frac{1}{(z-w)}\Big) :\partial b(z)\partial c(w):$$
$$+ \Big(-\frac{2}{\pi g}\frac{1}{(z-w)^3}\Big) :c(z) b(w): \Big]$$
$$+ \Big(\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big)\Big(-\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big) + \Big(\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big) :\partial b(z) c(w):$$
$$+ \Big(-\frac{1}{\pi g}\frac{1}{(z-w)^2}\Big) : c(z)\partial b(w):$$

$$= -4\frac{1}{(z-w)^4} - 4\pi g \frac{:b(z)\partial c(w):}{(z-w)^2} + 4\pi g \frac{:\partial c(z)\, b(w):}{(z-w)^2}$$

$$-4\frac{1}{(z-w)^4} - 4\pi g \frac{:b(z)c(w):}{(z-w)^3} + 2\pi g \frac{:\partial c(z)\partial b(w):}{(z-w)}$$

$$-4\frac{1}{(z-w)^4} + 2\pi g \frac{:\partial b(z)\partial c(w):}{(z-w)} - 4\pi g \frac{:c(z)b(w):}{(z-w)^3}$$

$$-\frac{1}{(z-w)^4} + \pi g \frac{:\partial b(z)c(w):}{(z-w)^2} - \pi g \frac{:c(z)\partial b(w):}{(z-w)^2}$$

$$= -13\frac{1}{(z-w)^4} + \frac{\pi g}{(z-w)^2}\Big\{ :\partial b(z)c(w): - :c(z)\partial b(w): + 4:\partial c(z)b(w): - 4:b(z)\partial c(w): \Big\}$$

$$+ 2\pi g \Big\{ \frac{:\partial b(z)\partial c(w):}{(z-w)} + \frac{:\partial c(z)\partial b(w):}{(z-w)} \Big\}$$

$$-4\pi g \frac{1}{(z-w)^3}\Big\{ :b(z)c(w): + :c(z)b(w): \Big\}$$

Taylor at $z=w$

$$= -13\frac{1}{(z-w)^4} + \frac{\pi g}{(z-w)^2}\Big\{ :\partial b(w)c(w): + :\partial_w^2 b(w)c(w):(z-w) - :c(w)\partial b(w): - :\partial c(w)\partial b(w):(z-w) + 4:\partial c(w)b(w): + 4:\partial_w^2 c(w)b(w):(z-w) - 4:b(w)\partial c(w): - 4:\partial b(w)\partial c(w):(z-w)$$

$$+ \frac{2\pi g}{(z-w)}\Big\{ :\partial b(w)\partial c(w): + :\partial c(w)\partial b(w): \Big\}$$

$$- \frac{4\pi g}{(z-w)^3}\Big\{ :b(w)c(w): + :\partial b(w)c(w):(z-w) + \frac{1}{2}:\partial_w^2 b(w)c(w):(z-w)^2 + :c(w)b(w): + :\partial c(w)b(w):(z-w) + \frac{1}{2}:\partial_w^2 c(w)b(w):(z-w)^2$$

+ reg.

$$= \frac{-13}{(z-w)^4} + \frac{\pi g}{(z-w)^2}\Big\{ :\partial b(w)c(w): - :c(w)\partial b(w): + 4:\partial c(w)b(w): - 4:b(w)\partial c(w): - 4:\partial b(w)c(w): - 4:\partial c(w)b(w):$$

$$+ \frac{\pi g}{(z-w)}\Big\{ :\partial^2 b(w)c(w): - :\partial c(w)\partial b(w): + 4:\partial^2 c(w)b(w): - 4:\partial b(w)\partial c(w): + 2:\partial b(w)\partial c(w): - 2:\partial^2 b(w)c(w): - 2:\partial^2 c(w)b(w): + 2:\partial c(w)\partial b(w): \Big\}$$

$$- \frac{4\pi g}{(z-w)^3}\Big\{ :b(w)c(w): + :c(w)b(w): \Big\} + \text{reg}.$$

$$= \frac{-13}{(z-w)^4} + \frac{\pi g}{(z-w)^2}\left\{-2 :\partial b(w)\, c(w): -4 :b(w)\partial c(w):\right\}$$

b and c anti-commute

$$+ \frac{\pi g}{(z-w)}\left\{- :\partial^2 b(w)\, c(w): -3 :\partial b(w)\partial c(w): + 2 :\partial^2 c(w)\, b(w): \right\} + \text{reg}.$$

why :AB: = - :BA: possible?

$$= \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg}.$$

once more, anticom.

$$2T(w) \sim 2\pi g\,(2 :\partial c b: + :c\partial b:)$$

$$\sim \pi g\,(2 :\partial^2 c b: + 2 :\partial c \partial b: + :\partial c \partial b: + :c\partial^2 b:)$$

$$\leadsto \quad T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{h\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg}$$

with $c = 26$ central charge

and $h = 2$ conformal weight