Disclaimer

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https://www.physics-and-stuff.com/

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String Theory Sheet 8 Homework

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1) $V_\alpha(z,\bar{z}) = :e^{i\alpha\phi(z,\bar{z})}:$; ϕ free bosonic field

Then, we calculate

$$:e^{a\phi_1}::e^{b\phi_2}: = :\sum_{n=0}^{\infty}\frac{(a\phi_1)^n}{n!}::\sum_{m=0}^{\infty}\frac{(b\phi_2)^m}{m!}:$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a^n b^m}{n!m!}:\phi_1^n::\phi_2^m:$$

Look at $:\phi_1^n::\phi_2^m:$ for a fixed n,m (contractions)

$$:\phi_1^n::\phi_2^m: = \sum_{k=0}^{\min\{n,m\}}\binom{n}{k}\cdot\frac{m!}{(m-k)!}\langle\phi_1\phi_2\rangle^k:\phi_1^{n-k}\phi_2^{m-k}:$$

$k=0$ is the purely normal ordered part (so it's included)

choose k out of n ϕ_1's to contract with ϕ_2's

the first ϕ_1 can be contracted with m ϕ_2's, the second w/ $(m-1)$ and so on. Interchanged combinations are different contractions $\Rightarrow$ not $\binom{m}{k}$

choose k to contract $\to$ prop.

remaining

- Note that we can let the sum run to ∞, because

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-k+1)}{k!} = 0 \text{ for } k>n$$

$\leftarrow$ one term $=0$

and $\frac{m!}{(m-k)!} = m\cdot(m-1)\cdot\ldots\cdot(m-k+1) = 0$ for $k>m$

Interchanging the sums and applying this argument again for the outer sums, they only run from k on, s.t.

Really the same argument again or why outer sum only from k on? / $\uparrow k \le n,m$

Rewrite $\sum_{n,m=0}$ to $\sum_{k\le n,m}$

$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}\frac{a^n b^m}{n!m!}\frac{n!}{(n-k)!k!}\frac{m!}{(m-k)!}\langle\phi_1\phi_2\rangle^k:\phi_1^{n-k}\phi_2^{m-k}:$$

$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}\frac{a^{n-k}}{(n-k)!}\frac{b^{m-k}}{(m-k)!}:\phi_1^{n-k}\phi_2^{m-k}:\frac{a^k b^k}{k!}\langle\phi_1\phi_2\rangle^k$$

Prop. commutes w/ $:\phi_1^{n-k}\phi_2^{m-k}:$ ✓ $\rightsquigarrow$ yes, only a c-number & (function)

Sum shift in n,m

$$= :\sum_{n=0}^{\infty}\frac{(a\phi_1)^n}{n!}\sum_{m=0}^{\infty}\frac{(b\phi_2)^m}{m!}:\sum_{k=0}^{\infty}\frac{(ab\langle\phi_1\phi_2\rangle)^k}{k!}$$

$$= :e^{a\phi_1}e^{b\phi_2}:e^{ab\langle\phi_1\phi_2\rangle}$$

bosons commute

$$= :e^{a\phi_1+b\phi_2}:e^{ab\langle\phi_1\phi_2\rangle}$$ ✓

For $V_\alpha(z,\bar z)$, we then find:

$$V_\alpha(z,\bar z)V_\beta(w,\bar w) = :e^{i\alpha\phi(z,\bar z)}::e^{i\beta\phi(w,\bar w)}:$$

$$= :e^{i\alpha\phi(z,\bar z)+i\beta\phi(w,\bar w)}: e^{-\alpha\beta\langle\phi(z,\bar z)\phi(w,\bar w)\rangle}$$

With $\langle\phi(z,\bar z)\phi(w,\bar w)\rangle = -\frac{1}{4\pi g}\log|z-w|^2 + c$

we find $e^{-\alpha\beta\langle\phi(z,\bar z)\phi(w,\bar w)\rangle} = e^{-\alpha\beta(-\frac{1}{2\pi g}\log|z-w|+c)}$

$$= |z-w|^{\frac{\alpha\beta}{2\pi g}} e^{-\alpha\beta c}$$

$$= :e^{i\alpha\phi(z,\bar z)+i\beta\phi(w,\bar w)}: |z-w|^{\frac{\alpha\beta}{2\pi g}} e^{-\alpha\beta c}$$

✓ Is this $:V_\alpha V_\beta:$, i.e. $:e^{a\phi_1}::e^{b\phi_2}:$? $:e^{a\phi_1}e^{b\phi_2}:$? No, exactly what we prove here ✓

What about the constant in the result? just norm. const.

Calculating $\langle V_\alpha(z,\bar z)V_\beta(w,\bar w)\rangle$, we will use that $\langle:\phi\ldots:\rangle=0$ and taylor $e^{i\beta\phi(w,\bar w)}$ at $(w,\bar w)=(z,\bar z)$:

Difference $\langle\ldots\rangle$ and $\langle 0|\ldots|0\rangle$?

$$\langle V_\alpha(z,\bar z)V_\beta(w,\bar w)\rangle = |z-w|^{\frac{\alpha\beta}{2\pi g}}\langle :e^{i\alpha\phi(z,\bar z)}\Big(e^{i\beta\phi(z,\bar z)} + \partial e^{i\beta\phi(z,\bar z)}(w-z) + \frac{\partial^2}{2}e^{i\beta\phi(z,\bar z)}(w-z)^2 + \ldots\Big):\rangle$$

$$= |z-w|^{\frac{\alpha\beta}{2\pi g}}\langle :e^{i(\alpha+\beta)\phi(z,\bar z)}: + :e^{i\alpha\phi(z,\bar z)}\partial e^{i\beta\phi(z,\bar z)}(w-z): \text{"higher orders"}:\rangle$$

$$= \begin{cases} |z-w|^{-\frac{\alpha^2}{2\pi g}} & \text{for } \alpha=-\beta \\ 0 & \text{else}\end{cases}$$

Still a 1 in $\exp(i(\alpha+\beta)\ldots)$ for $\alpha\neq-\beta$?

2) Recall (1.10) OPE from sheet 7

1.
$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{finite terms}$$

We had

$$\delta_\epsilon T(w) = -[Q_\epsilon, T(w)]$$
$$= -\left[\oint_{C_0} \frac{dz}{2\pi i} \epsilon(z)T(z), T(w)\right]$$
$$= -\oint_{C_0} \frac{dz}{2\pi i} [\epsilon(z)T(z), T(w)]$$

sheet 6
$$= -\oint_{C_w} \frac{dz}{2\pi i} R(\epsilon(z)T(z)T(w))$$
$$= -\oint_{C_w} \frac{dz}{2\pi i} \epsilon(z) \left\{ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg.} \right\}$$

C.R. sheet 7
$$= -\left\{ \frac{c}{12}\partial_w^3 \epsilon(w) + 2T(w)\partial_w \epsilon(w) + \epsilon(w)\partial_w T(w) \right\}$$

What about the R? Can't pull $\epsilon(z)$ out of the R, but only have $T(z)T(w)$? gives the same, as $\epsilon(z)$ just a perturbation and $\mathbb{1}$ in operator space $\epsilon(z)\otimes\mathbb{1}$ instead of a_i and a_i.

$$\Rightarrow \delta_\epsilon T(z) = -\epsilon(z)\partial_z T(z) - 2\partial_z \epsilon(z)T(z) - \frac{c}{12}\partial_z^3 \epsilon(z)$$

How to get finite version?

2. The Schwarzian derivative is defined as
$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2 \quad \text{where } w' = \partial_z w$$

If charge $c=0$, would be primary?

Look at $w = \frac{az+b}{cz+d}$ with $ad-bc=1$ because of the group $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{Z}_2$

only for this w?

$$\Rightarrow w' = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-cb}{(cz+d)^2}$$
$$= \frac{1}{(cz+d)^2} = (cz+d)^{-2}$$

$$w'' = -2(cz+d)^{-3}\cdot c = -2c(cz+d)^{-3}$$
$$w''' = 6c^2(cz+d)^{-4}$$

Difference (quasi) primary field, (only) global / local trafo behavior; global in different sense?

$$\Rightarrow \frac{w'''}{w'} = 6c^2(cz+d)^{-2}$$
$$\frac{3}{2}\left(\frac{w''}{w'}\right)^2 = \frac{3}{2}\left(-2c(cz+d)^{-1}\right)^2 = 6c^2(cz+d)^{-2}$$

$\Rightarrow \{w,z\} = 0$ and thus

$$T(z) \to T(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) \quad \text{, i.e. EMT}$$

is a quasi primary field w/ conf. dim. $h = 2$

3. $z = e^{\frac{2\pi}{\ell} w} = e^{\frac{2\pi}{\ell}(\tau - i\sigma)}$, $\bar{z} = e^{\frac{2\pi}{\ell}\bar{w}} = e^{\frac{2\pi}{\ell}(\tau + i\sigma)}$

$\rightsquigarrow w = \frac{\ell}{2\pi} \log z$, $\bar{w} = \frac{\ell}{2\pi} \log \bar{z}$

$\rightsquigarrow w' = \frac{\ell}{2\pi z}$, $w'' = -\frac{\ell}{2\pi z^2}$, $w''' = \frac{\ell}{\pi z^3}$ and we find

$$T_{cyl}(w) = \left(\frac{dw}{dz}\right)^{-2} \left(T_{plane}(z) - \frac{c}{12}\{w,z\}\right)$$

Why Wick rot.?

$$= \left(\frac{\ell}{2\pi z}\right)^{-2} \left\{ T_{plane}(z) - \frac{c}{12}\left(\frac{\ell}{\pi z^3} \frac{2\pi z}{\ell} - \frac{3}{2}\left(-\frac{\ell}{2\pi z^2} \frac{2\pi z}{\ell}\right)^2 \right) \right\}$$

$$= \left(\frac{2\pi}{\ell}\right)^2 z^2 \left\{ T_{plane}(z) - \frac{c}{12}\left(\frac{2}{z^2} - \frac{3}{2}\frac{1}{z^2} \right) \right\}$$

$$= \left(\frac{2\pi}{\ell}\right)^2 z^2 \left\{ T_{plane}(z) - \frac{c}{24 z^2} \right\} = \left(\frac{2\pi}{\ell}\right)^2 \left\{ z^2 T_{plane}(z) - \frac{c}{24} \right\}$$

4. $$E = \int_0^\ell d\sigma \langle T_{\tau\tau} \rangle = -\frac{1}{2\pi} \int_0^\ell d\sigma \left(\langle T_{cyl}(w) \rangle + \langle \bar{T}_{cyl}(\bar{w}) \rangle \right)$$

Why $\langle T_{plane}(z) \rangle = 0$?

$$= -\frac{1}{2\pi} \int_0^\ell d\sigma \left\{ \left(\frac{2\pi}{\ell}\right)^2 \left[\langle T_{plane}(z) \rangle - \frac{c}{24} \right] + \left(\frac{2\pi}{\ell}\right)^2 \left[\langle \bar{T}_{plane}(\bar{z}) - \frac{c}{24} \right] \right\}$$

Why $\bar{T}_{cyl}(\bar{w})$ and how to get it from $T_{cyl}(w)$?

$$= \frac{1}{2\pi} \cdot 2 \left(\frac{2\pi}{\ell}\right)^2 \cdot \ell \cdot \frac{c}{24} = \frac{4\pi}{\ell} \frac{c}{24} = \frac{\pi c}{6\ell}$$

$$= \frac{c}{24\pi} R(x) \quad \text{w/} \quad R(x) = \frac{4\pi^2}{\ell}$$

For ~~[illegible]~~ Euclidean plane,

$\langle T \rangle \sim R = 0$

No curvature on cylinder?

Cylinder does not have intrinsic curvature but it does have the extrinsic one.

And the ~~stree~~ nonvanishing T arises because of a finite size of L.