Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

String Theory Exercise 9 Homework   Marvin Zanke

05.12.2018

1) $L_{-k_1}L_{-k_2}\dots L_{-k_n}|h\rangle$ ⇝ Gram matrix

$$M = \begin{pmatrix} M^{(1)} & & & \\ & M^{(2)} & & \\ & & M^{(3)} & \\ & & & \ddots \end{pmatrix}$$

$\ell=0$: $|h\rangle$

$\ell=1$: $L_{-1}|h\rangle$

$\ell=2$: $L_{-1}L_{-1}|h\rangle$ and $L_{-2}|h\rangle$

⇝ $\ell=0$: $M^{(0)} = \langle h|h\rangle = 1$

$L_1|h\rangle = 0$

$\ell=1$: $M^{(1)} = \langle h|L_1L_{-1}|h\rangle = \langle h|[L_1,L_{-1}]|h\rangle$

$= \langle h|2L_0 + \frac{c}{12}(1^3-1)|h\rangle = 2h$

$\ell=2$: $M^{(2)} = \begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_2L_{-1}L_{-1}|h\rangle \\ \langle h|L_1L_1L_{-2}|h\rangle & \langle h|L_1L_1L_{-1}L_{-1}|h\rangle \end{pmatrix}$

$M^{(2)}_{11} = \langle h|L_2L_{-2}|h\rangle = \langle h|4L_0 + \frac{c}{12}(2^3-2)|h\rangle$

$= 4h + \frac{c}{2}$

$M^{(2)}_{12} = \langle h|L_2L_{-1}L_{-1}|h\rangle = \langle h|3L_1L_{-1} + L_{-1}L_2L_{-1}|h\rangle$

$= \langle h|3L_1L_{-1} + L_{-1}(3L_1 + L_{-1}L_2)|h\rangle$

$= 3\langle h|L_{-1}L_1 + 2L_0 + \frac{c}{12}(1^3-1)|h\rangle$

$M^{(2)\dagger}_{21}$ or $M^{(2)}_{21}$?

$M = M^\dagger$ (for $M_{ij} = \langle i|j\rangle$)

$= 6h = M^{(2)\dagger}_{21}$

$M^{(2)}_{22} = \langle h|L_1L_1L_{-1}L_{-1}|h\rangle = \langle h|L_1(2L_0)L_{-1}$
$+ L_1L_{-1}L_1L_{-1}|h\rangle$

$= \langle h|2L_1(L_{-1}L_0 + L_{-1}) + L_1L_{-1}(L_{-1}L_1 + 2L_0)|h\rangle$

$= \langle h|2(L_{-1}L_1 + 2L_0)(h+1) + 2h(L_{-1}L_1 + 2L_0)|h\rangle$

$= \langle h|4h(h+1) + 4h^2|h\rangle = 8h^2 + 4h$

$$\rightsquigarrow M^{(2)} = \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 8h^2 + 4h \end{pmatrix}$$

A general state can be spanned by

$$\chi = \sum_i c_i |\chi_i\rangle \quad \text{with} \quad |\chi_i\rangle = L_{-k_1^i} \dots L_{-k_{n_i}^i} |h\rangle$$

Then $\langle \chi | \chi \rangle = \sum_{ij} c_i^* c_j \langle \chi_i | \chi_j \rangle = \sum_{ij} c_i^* c_j M_{ij}$

diagonalize

$$= \sum_{ijkl} c_i^* c_j U_{ik}^* \lambda_k \delta_{kl} U_{lj} = \sum_k |(Uc)_k|^2 \lambda_k$$

$\rightsquigarrow$ negative norm states iff M has one or more negative eigenvalues. and we know $\det M^{(1)} = 2h \rightsquigarrow h > 0$

2) A singular vector is a null state, i.e. orthogonal and norm 0.

$$\det M^{(2)} = \left(4h + \frac{c}{2}\right)\left(8h^2 + 4h\right) - 36h^2$$
$$= 32h^3 + 16h^2 + 4h^2 c + 2hc - 36h^2$$
$$= 32h^3 - 20h^2 + 4h^2 c + 2hc$$

Mathematica (roots) $\rightarrow$
$$= 32h\left(h - \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right)\right) \times \left(h - \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right)\right) \stackrel{!}{=} 0$$

$$\Rightarrow h = \frac{1}{16}\left(5 - c \pm \sqrt{(1-c)(25-c)}\right) \quad (h \neq 0 \text{ given})$$

4) Consider $|\chi\rangle = [L_{-2} + \eta L_{-1}^2]|h\rangle$

$$L_1|\chi\rangle = 3L_{-1} + \eta\left(L_{-1}L_1 + 2L_0 + \frac{c}{12}(1^3-1)\right)L_{-1}|h\rangle$$
$$= 3L_{-1} + \eta\{L_{-1}(L_{-1}L_1 + 2L_0) + 2(L_{-1}L_0 + L_{-1})\}|h\rangle$$
$$= (3 + 2\eta h + 2\eta h + 2\eta)L_{-1}|h\rangle$$
$$= (3 + 2\eta + 4\eta h)L_{-1}|h\rangle$$

$$L_2|\chi\rangle = 4L_0 + \frac{c}{12}(2^3-2) + \eta(L_{-1}L_2 + 3L_1)L_{-1}|h\rangle$$
$$= 4L_0 + \frac{c}{2} + \eta L_{-1}(L_{-1}L_2 + 3L_1) + 3\eta(L_{-1}L_1 + 2L_0)|h\rangle$$
$$= 4h + \frac{c}{2} + 6h\eta\,|h\rangle$$

We already found that for $|\chi\rangle$ to be singular, we need

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right)$$

For $|\chi\rangle$ to be a nullstate, we also need

$$L_1|\chi\rangle = L_2|\chi\rangle = 0$$

From the first one, we find

$$3 + 2\eta(1+2h)L_{-1}|h\rangle = 0$$
$$\Rightarrow \eta = \frac{-3}{2(2h+1)}$$

For the nullstate $|\chi\rangle$, we then associate a descendent null field $\chi(z)$ (descendent of the primary field $\phi(z)$ of conformal dimension h; itself primary field of dimension $h+2$).

$h>?$ $\quad |h\rangle = \lim_{z\to 0} \phi_h(z)|0\rangle$

what for this step with $|h\rangle = ..?$ $\quad |\chi\rangle = \chi(z)|0\rangle = [L_{-2} + \eta L_{-1}^2]\underbrace{\phi(z)|0\rangle}_{=|h\rangle}$

Why $\phi^{(-2)}(z)$?

$$= [\phi^{(-2)}(z) + \eta\,\phi^{(-1,-1)}(z)]|0\rangle$$

$L_{-1}L_{-1}\phi(z) = L_{-1}\phi^{(-1)}(z) = L_{-1}\partial\phi(z) = \partial^2\phi(z)$

using residue theorem and expansion (OPE) for $T(z)\phi(w)$

$$= \phi^{(-2)}(z) + \eta\, \partial^2 \phi(z) = \phi^{(-2)}(z) - \frac{3}{2(2h+1)} \underbrace{\frac{\partial^2}{\partial z^2}\phi(z)}_{\mathcal{L}_{-1}^2 \phi(z)}$$

For $X = \phi_1(w_1) \dots \phi_N(w_N)$, we also find for the product with the null state (orthogonal) that

$$\langle \chi(z)\, X \rangle = \langle 0 | \chi(z)\, X | 0 \rangle = \langle \chi | X \rangle = 0$$

$$= \underbrace{\langle \phi^{(-2)}(z)\, X \rangle}_{\mathcal{L}_{-2} \langle \phi(z)\, X \rangle} - \frac{3}{2(2h+1)} \mathcal{L}_{-1} \langle \phi(z)\, X \rangle$$

Why now invert what we found for $\mathcal{L}_{-1}, \mathcal{L}_{-2}$?

$$\rightarrow \left(\mathcal{L}_{-2} - \frac{3}{2(2h+1)} \mathcal{L}_{-1}^2 \right) \langle \phi(z)\, X \rangle = 0$$

$$\Leftrightarrow \left\{ \sum_{i=1}^{N} \left[\frac{1}{z - z_i} \frac{\partial}{\partial z_i} + \frac{h_i}{(z - z_i)^2} \right] - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \right\} \langle \phi(z)\, X \rangle = 0$$

Consider the three point function $X = \phi_1(z_1)\phi_2(z_2)$ with

$$\langle \phi(z)\, \phi_1(z_1)\, \phi_2(z_2) \rangle = \frac{C_{123}}{(z - z_1)^{h_2 - h - h_1} (z_1 - z_2)^{h - h_1 - h_2} (z - z_2)^{h_1 - h - h_2}}$$