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Theoretical Particle Physics Homework 1

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1)  $t' = \gamma(t - vx')$

1.  $x'^1 = \gamma(-vt + x^1)$

and

$x'^2 = x^2$

$x'^3 = x^3$

where  $\gamma^{-1} = \sqrt{1 - v^2}$ ,  $c=1$

We have 
$$\begin{pmatrix} t' \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

As the inverse to  $x'^3$  and  $x'^2$  is trivial, we can reduce the problem to inverting  $A = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}$

$$\rightarrow \left( \begin{array}{cc|cc} \gamma & -\gamma v & 1 & 0 \\ -\gamma v & \gamma & 0 & 1 \end{array} \right) \xrightarrow{1+2} \left( \begin{array}{cc|cc} \gamma - \gamma v^2 & 0 & 1 & v \\ -\gamma v & \gamma & 0 & 1 \end{array} \right)$$

$$\xrightarrow{\cdot 1/\gamma} \left( \begin{array}{cc|cc} \gamma(1-v^2) & 0 & 1 & v \\ -\gamma v & \gamma & 0 & 1 \end{array} \right) \xrightarrow{1+2} \left( \begin{array}{cc|cc} 1/\gamma & 0 & 1 & v \\ 0 & \gamma & \gamma^2 v & 1+\gamma^2 v^2 \end{array} \right)$$

$$\xrightarrow{\begin{matrix} \cdot \gamma \text{ in } 1 \\ \cdot \gamma \text{ in } 2 \end{matrix}} \left( \begin{array}{cc|cc} 1 & 0 & \gamma & \gamma v \\ 0 & 1 & \gamma v & \frac{1+\gamma^2 v^2}{\gamma} \end{array} \right), \quad \frac{1+\gamma^2 v^2}{\gamma} = \frac{1}{\gamma} \left( 1 + v^2 \frac{1}{1-v^2} \right)$$

$A^{-1}$

$$= \frac{1}{\gamma} \left( 1 - \frac{v^2 - 1 + 1}{v^2 - 1} \right) = \gamma$$

$$\rightarrow A^{-1} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix}$$

2.  $\frac{\partial \phi}{\partial x^r}$  is covariant, as:  $x^r \frac{\partial \phi}{\partial x^r} \xrightarrow{\Lambda} x'^\mu \frac{\partial \phi}{\partial x'^\mu}$

and 
$$x'^\mu \frac{\partial \phi}{\partial x'^\mu} = x'^r \frac{\partial \phi}{\partial x^r} \frac{\partial x^k}{\partial x'^\mu} = x'^\mu \frac{\partial \phi}{\partial x^k} \left( \Lambda^{-1} \right)^k_\mu$$

$$= x^k \frac{\partial \phi}{\partial x^k}$$

$$\Rightarrow \frac{\partial \phi}{\partial x^k} = \partial_k \phi$$

$$\begin{aligned} x'^\nu &= \Lambda^\nu_\mu x^\mu \\ \Leftrightarrow x^1 &= (\Lambda^{-1})^1_\nu x'^\nu \\ \Rightarrow \frac{\partial \phi}{\partial x^k} &= (\Lambda^{-1})^k_\nu \frac{\partial \phi}{\partial x'^\nu} \end{aligned}$$

$$2] \quad \gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \sigma^i = \text{Pauli-matrices}$$

We want to show that this representation (Dirac repr.)

fulfills the Clifford algebra  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\underline{\mu, \nu=0}: \quad \gamma^0 \gamma^0 + \gamma^0 \gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} + \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} = 2\mathbb{1}_4 = 2g^{00}\mathbb{1}_4 \checkmark$$

$$\left( \begin{array}{l} \mu=0, \nu=i: \\ \stackrel{\uparrow}{\sim} \mu=i, \nu=0 \\ \text{as } \{\gamma^\mu, \gamma^\nu\} = \{\gamma^\nu, \gamma^\mu\} \end{array} \right) \quad \gamma^0 \gamma^i + \gamma^i \gamma^0 = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0 = 2g^{0i}\mathbb{1}_4 \checkmark$$

$\uparrow 0$

$$\underline{\mu=i, \nu=j}: \quad \gamma^i \gamma^j + \gamma^j \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^i \sigma^j & 0 \\ 0 & -\sigma^i \sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j \sigma^i & 0 \\ 0 & -\sigma^j \sigma^i \end{pmatrix}$$

$$= - \begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} \stackrel{\uparrow \text{Literature}}{=} -2\delta_{ij} \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

$$= 2g^{ij}\mathbb{1}_4 \checkmark$$

$\uparrow -\delta_{ij}$

Prove this  $\{\sigma^i, \sigma^j\}$  identity?  $\checkmark$   
 $\rightarrow$  better learn by heart

3] Dirac equation:  $i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad (*)$

$\psi = u e^{-ipx}$ ,  $p^\mu = (p^0, \vec{p})$ ,  $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = N \begin{pmatrix} \phi \\ \chi \end{pmatrix}$   
 $\uparrow$   
 $\in \mathbb{C}^2$ , two-comp spinors

1.  $\bar{\psi}(x) \equiv \psi^\dagger(x) \gamma^0$

$(*)^\dagger \Leftrightarrow -i \partial_\mu \psi^\dagger (\gamma^\mu)^\dagger - m\psi^\dagger = 0$

$\partial_\mu^\dagger = \partial_\mu$

$\Leftrightarrow -i \partial_\mu \overbrace{\psi^\dagger}^{\bar{\psi}} \gamma^0 \gamma^\mu \gamma^0 - m\psi^\dagger = 0$

$\Leftrightarrow -i \partial_\mu \bar{\psi} \gamma^\mu (\gamma^0)^2 - m\psi^\dagger \gamma^0 = 0$   
 $\times (\gamma^0)$

$\Leftrightarrow -i \partial_\mu \bar{\psi} \gamma^\mu - m\bar{\psi} = 0 \Leftrightarrow \bar{\psi} (i\overleftarrow{\partial}_\mu \gamma^\mu + m) = 0$

2.  $j^\mu = \bar{\psi} \gamma^\mu \psi$

$0 = \partial_\mu j^\mu = \partial_\mu (\bar{\psi} \gamma^\mu \psi) = \bar{\psi} \overleftarrow{\partial}_\mu \psi + \bar{\psi} \overrightarrow{\partial}_\mu \psi$   
 $= im\bar{\psi}\psi - im\bar{\psi}\psi = 0 \quad \checkmark$

$i\bar{\psi} \overleftarrow{\partial}_\mu \psi = -m\bar{\psi}$   
 and  $i\psi \overrightarrow{\partial}_\mu \psi = m\psi$

3.  $N = \sqrt{E+m}$ ,  $u = N \begin{pmatrix} \phi \\ \frac{\vec{p} \cdot \vec{\sigma}}{E+m} \phi \end{pmatrix}$ ,  $v = \begin{pmatrix} \frac{\vec{p} \cdot \vec{\sigma}}{E-m} \chi \\ \chi \end{pmatrix}$   
 $U U^\dagger = U^\dagger \gamma^0 U = (E+m) \begin{pmatrix} \phi^\dagger & \phi^\dagger \frac{\vec{p} \cdot \vec{\sigma}}{E+m} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi \\ \frac{\vec{p} \cdot \vec{\sigma}}{E+m} \phi \end{pmatrix}$   
 $\stackrel{\vec{p}^\dagger = \vec{p}}{\sigma_j^\dagger = \sigma_j} \downarrow = \begin{pmatrix} \phi^\dagger & \phi^\dagger \frac{\vec{p} \cdot \vec{\sigma}}{E+m} \end{pmatrix} \begin{pmatrix} \phi \\ -\frac{\vec{p} \cdot \vec{\sigma}}{E+m} \phi \end{pmatrix} (E+m)$

$\sigma_j^\dagger = \sigma_j$  in every representation?  
 $\rightarrow$  Every repr.

Why don't we have to take care of an additional "u" like in  $V_{\vec{p}} u = u \sigma_2 \dots \sigma_2 \uparrow^2$   
 $\rightarrow$  Spinor space,  $\uparrow^2$ ?  
 No minimalistic vector

$= (E+m) \left\{ \phi^\dagger \phi - \phi^\dagger \frac{(\vec{p} \cdot \vec{\sigma})^2}{(E+m)^2} \phi \right\}$ ,  $p_i \sigma_i p_j \sigma_j = \frac{1}{2} p_i p_j \{\sigma_i, \sigma_j\} = |\vec{p}|^2$   
 $= (E+m) \phi^\dagger \phi \left\{ 1 - \frac{|\vec{p}|^2}{(E+m)^2} \right\} = (E+m) \phi^\dagger \phi \left\{ \frac{E^2 + m^2 + 2Em - |\vec{p}|^2}{(E+m)^2} \right\}$

$\stackrel{E^2 = p^2 + m^2}{\downarrow} = \phi^\dagger \phi \frac{2m^2 + 2Em}{E+m} = \phi^\dagger \phi (2m) = 2m$

Can we choose  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ?  
 $\rightarrow$  Basis vectors

$= 1$  in  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  repr.

$$-\vec{\sigma} \cdot \vec{p} = -\sigma^x p_x = \begin{matrix} \uparrow \\ E > 0 \end{matrix} (E + m) \begin{pmatrix} \chi^+ & \frac{-p_x - i p_y}{E + m} \chi^+ \\ \frac{-p_x + i p_y}{E + m} \chi^+ & -\chi^+ \end{pmatrix}$$

$$= -(E + m) \chi^+ \chi^+ \left\{ \frac{\vec{p}^2}{(E + m)^2} - 1 \right\} = -(E + m) \chi^+ \chi^+ \left\{ \frac{\vec{p}^2 - (E^2 + m^2 + 2Em)}{(E + m)^2} \right\}$$

$\uparrow$  use  $E > 0$

$$= -\frac{-2m^2 - 2Em}{E + m} = 2m$$

4.  $Z_C = i\gamma^2 Z_C^*$  First of all, we calculate  $u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}$

explicitly:  $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\tilde{N} = \sqrt{|E| + m}, \quad \delta_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \phi^{(1)} = \chi^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi^{(2)} = \chi^{(1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rightarrow u^{(1,2)} = \tilde{N} \begin{pmatrix} \phi^{(1,2)} \\ \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \phi^{(1,2)} \end{pmatrix} = \tilde{N} \begin{pmatrix} \phi^{(1,2)} \\ \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \frac{\phi^{(1,2)}}{E + m} \end{pmatrix}$$

$$= \begin{cases} \tilde{N} \begin{pmatrix} 1 \\ \frac{1}{E + m} p_z \\ \frac{1}{E + m} (p_x + ip_y) \\ 0 \end{pmatrix}, & u^{(1)} \\ \tilde{N} \begin{pmatrix} 1 \\ \frac{1}{E + m} p_z \\ \frac{1}{E + m} (p_x - ip_y) \\ 0 \end{pmatrix}, & u^{(2)} \end{cases}$$

$$u^{(3,4)} = \tilde{N} \begin{pmatrix} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \frac{\chi^{(1,2)}}{E + m} \\ \chi^{(1,2)} \end{pmatrix} \begin{matrix} \uparrow \\ E < 0, \\ |E| \text{ for } \\ E < 0 \end{matrix} \begin{cases} \tilde{N} \begin{pmatrix} \frac{1}{E + m} (-p_z) \\ \frac{1}{E + m} (p_x - ip_y) \\ 0 \\ 1 \end{pmatrix}, & u^{(3)} \\ \tilde{N} \begin{pmatrix} \frac{1}{E + m} (-p_z) \\ \frac{1}{E + m} (p_x + ip_y) \\ 0 \\ 1 \end{pmatrix}, & u^{(4)} \end{cases}$$

Now:  $Z_C^{(1,2)} = i\gamma^2 (u^{(3,4)})^*$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} (u^{(3,4)})^* = \begin{cases} \tilde{N} \begin{pmatrix} \frac{1}{E + m} (p_x - ip_y) \\ \frac{1}{E + m} (-p_z) \\ 0 \\ +1 \end{pmatrix}, & u^{(1)}(p) \\ \tilde{N} \begin{pmatrix} \frac{1}{E + m} (-p_z) \\ \frac{1}{E + m} (p_x + ip_y) \\ -1 \\ 0 \end{pmatrix}, & u^{(2)}(p) \end{cases} \begin{cases} +u^{(1)}(-p) \\ -u^{(2)}(-p) \end{cases}$$

formal way to prove this w/o explicit representation → doesn't know a way.

Why do we always need  $|E|$  in order to get the correct factors → take  $\chi$  out of denominator,  $|E| + m$  in norm constant

$\vec{p} \mapsto -\vec{p}$ ?

Why  $u_C$  physically?  $v_C$  modes → make more sense?  $u_C$  w/ my energy  $\hat{=} e^+$  w/ pos. on