10.10.2017

Theoretical Particle Physics Homework 1

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1|

1. $t' = \gamma(t - v x^1) = \gamma t - \gamma v x^1$ and $x'^2 = x^2$

$x'^1 = \gamma(-vt + x^1) = -\gamma v t + \gamma x^1$ $\qquad x'^3 = x^3$

where $\gamma^{-1} = \sqrt{(1-v^2)}$, $c = 1$

We have
$$\begin{pmatrix} t' \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

As the inverse to x'^3 and x'^2 is trivial, we can reduce the problem to inverting $A = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}$

$$\rightsquigarrow \left(\begin{array}{cc|cc} \gamma & -\gamma v & 1 & 0 \\ -\gamma v & \gamma & 0 & 1 \end{array}\right) \overset{\cdot v}{\underset{2+1}{\rightsquigarrow}} \left(\begin{array}{cc|cc} \gamma - \gamma v^2 & 0 & 1 & v \\ -\gamma v & \gamma & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} \overbrace{\gamma(1-v^2)}^{1/\gamma} & 0 & 1 & v \\ -\gamma v & \gamma & 0 & 1 \end{array}\right) \overset{\cdot \gamma^2 v}{\underset{1+2}{\rightsquigarrow}} \left(\begin{array}{cc|cc} 1/\gamma & 0 & 1 & v \\ 0 & \gamma & \gamma^2 v & 1+\gamma^2 v^2 \end{array}\right)$$

$$\overset{\cdot \gamma \text{ in } 1}{\underset{:\gamma \text{ in } 2}{\rightsquigarrow}} \left(\begin{array}{cc|cc} 1 & 0 & \gamma & \gamma v \\ 0 & 1 & \gamma v & \frac{1+\gamma^2 v^2}{\gamma} \end{array}\right), \qquad \frac{1+\gamma^2 v^2}{\gamma} = \frac{1}{\gamma}\left(1 + v^2 \frac{1}{1-v^2}\right)$$

$$\underbrace{\hphantom{\gamma \quad \frac{1+\gamma^2 v^2}{\gamma}}}_{A^{-1}} \qquad = \frac{1}{\gamma}\left(1 - \frac{v^2 - 1 + 1}{v^2 - 1}\right)$$

$$\Rightarrow \gamma$$

$$\rightsquigarrow A^{-1} = \begin{pmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{pmatrix}$$

2. $\frac{\partial \phi}{\partial x^\mu}$ is covariant, as: $x^\mu \frac{\partial \phi}{\partial x^\mu} \xmapsto{\Lambda} x'^\mu \frac{\partial \phi}{\partial x'^\mu}$

and $x'^\mu \frac{\partial \phi}{\partial x'^\mu} = x'^\mu \frac{\partial \phi}{\partial x^k} \frac{\partial x^k}{\partial x'^\mu} = x'^\mu \frac{\partial \phi}{\partial x^k} (\Lambda^{-1})^k{}_\mu$

$= x^k \frac{\partial \phi}{\partial x^k}$

$$\Longrightarrow \frac{\partial \phi}{\partial x^k} = \partial_k \phi$$

$\uparrow$ $x'^\nu = \Lambda^\nu{}_\lambda x^\lambda$

$\Leftrightarrow x^\lambda = (\Lambda^{-1})^\lambda{}_\nu x'^\nu$

$\rightsquigarrow \frac{\partial x^\lambda}{\partial x'^\nu} = (\Lambda^{-1})^\lambda{}_\nu$

2| $\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, σ^i : Pauli-matrices

We want to show that this Representation (Dirac Reps.) fulfills the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$\mu, \nu = 0$: $$\gamma^0\gamma^0 + \gamma^0\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} + \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} = 2\mathbb{1}_4 = 2g^{00}\mathbb{1}_4 \checkmark$$

$\left(\begin{array}{l} \mu = 0, \nu = i : \\ \Leftrightarrow \mu = i, \nu = 0 \\ \text{as } \{\gamma^\mu, \gamma^\nu\} = \{\gamma^\nu, \gamma^\mu\} \end{array}\right)$ $$\gamma^0\gamma^i + \gamma^i\gamma^0 = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = 0 = 2\underset{\uparrow 0}{g^{0i}}\mathbb{1}_4 \checkmark$$

$\mu = i, \nu = j$: $$\gamma^i\gamma^j + \gamma^j\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$= -\begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} \underset{\substack{\uparrow \\ \text{Literature} \\ \{\sigma^i, \sigma^j\} = 2\delta_{ij}\mathbb{1}_2}}{=} -2\delta_{ij}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

$$= 2\underset{\uparrow -\delta_{ij}}{g^{ij}}\mathbb{1}_4 \checkmark$$

Prove this $\{\sigma^i, \sigma^j\}$ identity? ✓
⇝ better learn by heart

3| Dirac equation: $i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad (*)$

$$\psi = w e^{-ipx}, \quad p^\mu = (p^0, \vec{p}), \quad w = \begin{pmatrix} w^0 \\ w^1 \\ w^2 \\ w^3 \end{pmatrix} = N \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

$\uparrow$ $\in \mathbb{C}^2$, two-comp. Spinors

1. $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$

$$(*)^\dagger \underset{\substack{\uparrow \\ \partial_\mu^\dagger = \partial_\mu}}{\Leftrightarrow} -i\partial_\mu \psi^\dagger (\gamma^\mu)^\dagger - m\psi^\dagger = 0$$

$$\underset{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}{\Leftrightarrow} -i\partial_\mu \overbrace{\psi^\dagger \gamma^0}^{\bar{\psi}} \gamma^\mu \gamma^0 - m\psi^\dagger = 0$$

$$\underset{\times(\gamma^0)}{\Leftrightarrow} -i\partial_\mu \bar{\psi}\gamma^\mu(\gamma^0)^2 - m\psi^\dagger\gamma^0 = 0$$

$$\underset{(\gamma^0)^2 = \mathbb{1}}{\Leftrightarrow} -i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0 \Leftrightarrow \bar{\psi}(i\overleftarrow{\partial}_\mu \gamma^\mu + m) = 0$$

2. $j^\mu = \bar{\psi}\gamma^\mu\psi$

$$0 \overset{!}{=} \partial_\mu j^\mu = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = \bar{\psi}\overleftarrow{\cancel{\partial}}\psi + \bar{\psi}\overrightarrow{\cancel{\partial}}\psi$$

$$= im\bar{\psi}\psi - im\bar{\psi}\psi = 0 \quad ✓$$

$\uparrow$

$i\bar{\psi}\overleftarrow{\partial}_\mu\gamma^\mu = -m\bar{\psi}$

and $i\gamma^\mu\overrightarrow{\partial}_\mu\psi = m\psi$

3. $N = \sqrt{E+m}$, $u = N\begin{pmatrix} \phi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi \end{pmatrix}$, $v = \begin{pmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E-m}\chi \\ \chi \end{pmatrix}$

$$\bar{u}u = u^\dagger\gamma^0 u = (E+m)\left(\phi^\dagger, \phi^\dagger\frac{(\vec{p}\cdot\vec{\sigma})^\dagger}{E+m}\right)\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\begin{pmatrix} \phi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi \end{pmatrix}$$

$$\underset{\substack{\vec{p}^\dagger = \vec{p} \\ \vec{\sigma}^\dagger = \vec{\sigma}}}{=} \left(\phi^\dagger, \phi^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\right)\begin{pmatrix} \phi \\ -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi \end{pmatrix}(E+m)$$

$$= (E+m)\left\{\phi^\dagger\phi - \phi^\dagger\frac{(\vec{p}\cdot\vec{\sigma})^2}{(E+m)^2}\phi\right\}, \quad p_i\sigma_i p_j\sigma_j = \frac{1}{2}p_i p_j\{\sigma_i, \sigma_j\} = |\vec{p}|^2$$

$$= (E+m)\phi^\dagger\phi\left\{1 - \frac{|\vec{p}|^2}{(E+m)^2}\right\} = (E+m)\phi^\dagger\phi\left\{\frac{E^2+m^2+2Em-|\vec{p}|^2}{(E+m)^2}\right\}$$

$$\underset{E^2 = \vec{p}^2 + m^2}{=} \phi^\dagger\phi\,\frac{2m^2+2Em}{E+m} = \underbrace{\phi^\dagger\phi}_{=1 \text{ in } \binom{1}{0}, \binom{0}{1} \text{ repr.}}(2m) = 2m$$

✓ $\sigma^\dagger = \sigma$ in every representation?
→ every repr.

✓ Why don't we have to take care of an additional "−" like in $U_\mu U^\mu = U^{0\,2} - \vec{U}^2$
$\uparrow$?
→ Spinor space, no minkowski vector

✓ Can we choose $\binom{1}{0}$, $\binom{0}{1}$?
→ Basis vectors

$$-\bar{v}v = -v^\dagger\gamma^0 v \underset{E>0}{\overset{\uparrow}{=}} -(E+m)\left(\chi^\dagger \frac{-\vec{p}\cdot\vec{\sigma}}{|E|+m}, \chi^\dagger\right)\begin{pmatrix}\frac{-\vec{p}\cdot\vec{\sigma}}{|E|+m}\chi \\ -\chi\end{pmatrix}$$

$$= -(E+m)\chi^\dagger\chi\left\{\frac{\vec{p}^2}{(|E|+m)^2} - 1\right\} \underset{\text{we } E>0}{\overset{}{\uparrow=}} -(E+m)\chi^\dagger\chi\left\{\frac{\vec{p}^2-(E^2+m^2+2Em)}{(E+m)^2}\right\}$$

$$= -\frac{-2m^2-2Em}{E+m} = 2m$$

4. $\psi_c = i\gamma^2\psi^*$ First of all, we calculate $u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}$ explicitly: $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\tilde{N} = \sqrt{|E|+m}$, $\gamma_2 = \begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}$, $\phi^{(1)} = \chi^{(2)} = \begin{pmatrix}1\\0\end{pmatrix}$, $\phi^{(2)} = \chi^{(1)} = \begin{pmatrix}0\\1\end{pmatrix}$

$$\rightsquigarrow u^{(1/2)} = \tilde{N}\cdot\begin{pmatrix}\phi^{(1/2)} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi^{(1/2)}\end{pmatrix} = \tilde{N}\begin{pmatrix}\phi^{(1/2)} \\ \begin{pmatrix}p_z & p_x-ip_y\\ p_x+ip_y & -p_z\end{pmatrix}\frac{\phi^{(1/2)}}{E+m}\end{pmatrix}$$

$$= \begin{cases}\tilde{N}\begin{pmatrix}1\\0\\\frac{1}{E+m}p_z\\\frac{1}{E+m}(p_x+ip_y)\end{pmatrix}, & u^{(1)}\\ \tilde{N}\begin{pmatrix}0\\1\\\frac{1}{E+m}(p_x-ip_y)\\\frac{1}{E+m}(-p_z)\end{pmatrix}, & u^{(2)}\end{cases}$$

$$v^{(1/2)} = \tilde{N}\begin{pmatrix}-\begin{pmatrix}p_z & p_x-ip_y\\ p_x+ip_y & -p_z\end{pmatrix}\frac{\chi^{(1/2)}}{E+m} \\ \chi^{(1/2)}\end{pmatrix} \underset{\substack{E>0,\\ \text{use}\\ |E| \text{ for}\\ E<0}}{\overset{}{\uparrow=}} \begin{cases}\tilde{N}\begin{pmatrix}\frac{1}{E+m}(-p_z)\\\frac{1}{E+m}(-p_x-ip_y)\\1\\0\end{pmatrix}, & v^{(2)}\\ \tilde{N}\begin{pmatrix}\frac{1}{E+m}(-p_x+ip_y)\\\frac{1}{E+m}p_z\\0\\1\end{pmatrix}, & v^{(1)}\end{cases}$$

Now: $u_c^{(1/2)} = i\gamma^2\left(u^{(1/2)}\right)^*$

$$= \begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}\left(u^{(1/2)}\right)^* = \begin{cases}\tilde{N}\begin{pmatrix}\frac{1}{E+m}(p_x-ip_y)\\\frac{1}{E+m}(-p_z)\\0\\+1\end{pmatrix}, & u^{(1)}(p)\\ \tilde{N}\begin{pmatrix}\frac{1}{E+m}(-p_z)\\\frac{1}{E+m}(-p_x-ip_y)\\-1\\0\end{pmatrix}, & u^{(2)}(p)\end{cases} = \begin{cases}+v^{(1)}(-p)\\ -v^{(2)}(-p)\end{cases}$$

formal way to prove this w/o explicit representation? → doesn't know a way.

Why do we always need $|E|$ in order to get the correct factors? → take it out of denominator, $\sqrt{|E|+m}$ in norm constant

$\vec{p} \mapsto -\vec{p}$?

Why u_C physically? v_C makes much more sense? u_C w/ neg. energy ≙ e^+ w/ pos. en