Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

13.12.2017 **Theoretical Particle Physics Homework 10** Marvin Zanke

1) $\mathcal{L}_0 = \bar\Psi(x)\,(i\gamma^\mu\partial_\mu - m)\,\Psi(x)$

Do these N components represent spin like in N=4 for QED or do they furthermore contain spin?

$\Psi \mapsto \Psi' = U(x)\Psi$, $U(x) = \exp(ig\,\chi_a(x)T_a)$

$U(x)^\dagger U(x) = 1$

$\rightsquigarrow \mathcal{L}_0 \mapsto \bar{\mathcal{L}}_0 = \bar\Psi(x)\, e^{-ig\chi_a(x)T_a}\,(i\gamma^\mu\partial_\mu - m)\, e^{ig\chi_b(x)T_b}\,\Psi(x)$

obviously not invariant.

✓

T_a commute w/ γ-matrices? $\rightsquigarrow$ yes, different space

$D_\mu = \partial_\mu + igA_{\mu a}T_a$ and demand

$(D_\mu\Psi) \mapsto (D_\mu\Psi)' = D_\mu'\Psi' = D_\mu' U(x)\Psi \overset{!}{=} U(x) D_\mu \Psi$

✓

Why valid ∀ trafos if valid for infinit.? $\rightsquigarrow$ Lie-Algebra, enough to prove for infinitesimal

Consider infinitesimal trafos, $U(x) = 1 + ig\,\chi_a(x)T_a + \mathcal{O}(\chi_a^2)$

and find how $A_{\mu a} \longrightarrow A_{\mu a}' = A_{\mu a} + \delta A_{\mu a}$

Rather δA_μ and not D_μ defined by same trafo as Ψ?

$\rightsquigarrow D_\mu' U(x)\Psi \overset{!}{=} U(x) D_\mu \Psi$

$\Longleftrightarrow (\partial_\mu + igA_{\mu a}T_a + ig\,\delta A_{\mu a}T_a)(1 + ig\chi_b(x)T_b + \mathcal{O}(\chi_b^2))\Psi$

$\overset{!}{=} (1 + ig\chi_b(x)T_b + \mathcal{O}(\chi_b^2))(\partial_\mu + igA_{\mu a}T_a)\Psi$

✓

$\Longleftrightarrow \underline{(1+ig\chi_b(x)T_b)\partial_\mu\Psi} + ig(\partial_\mu\chi_b(x))T_b\Psi + \underline{igA_{\mu a}T_a\Psi} + ig\,\delta A_{\mu a}T_a\Psi$
$\quad - g^2 A_{\mu a}T_a\chi_b(x)T_b\Psi - \underbrace{g^2\delta A_{\mu a}T_a\chi_b(x)T_b\Psi}_{\sim\mathcal{O}(\delta A_{\mu a}\chi_b)} + \mathcal{O}(\chi_b^2)$

$\delta A_{\mu a}\chi_b \approx 0$? $\rightsquigarrow$ in the tut. we calculated this for A_μ' and then inserted the solution into itself again $\rightarrow A_\mu' T_a = \ldots + \ldots + C A_\mu^2 T_a T_b$ terms $+ \ldots$

$\overset{!}{=} \underline{(1+ig\chi_b(x)T_b)\partial_\mu\Psi} + \underline{igA_{\mu a}T_a\Psi} - g^2\chi_b(x)T_b A_{\mu a}T_a\Psi + \mathcal{O}(\chi_b^2)$

$\Longleftrightarrow ig(\partial_\mu\chi_b(x))T_b\Psi + ig\,\delta A_{\mu a}T_a\Psi - g^2 A_{\mu a}T_a\chi_b(x)T_b\Psi$
$\quad + g^2\chi_b(x)T_b A_{\mu a}T_a\Psi$
$\quad + \mathcal{O}(\chi_b^2) + \mathcal{O}(\chi_b\,\delta A_{\mu a}) = 0$

$\overset{\text{omitting higher order terms}}{\Longleftrightarrow} \delta A_{\mu a}T_a\Psi = -(\partial_\mu\chi_b(x))T_b\Psi + \frac{g}{i}\{A_{\mu a}\chi_b(x)T_aT_b - A_{\mu a}\chi_b(x)T_bT_a\}\Psi$

$\overset{\forall\Psi}{\Longleftrightarrow} \delta A_{\mu a}T_a = -(\partial_\mu\chi_a(x))T_a + \frac{g}{i}A_{\mu a}\chi_b(x)\,(if^{abc}T_c)$

$$\Leftrightarrow \delta A_{\mu a} T_a = -(\partial_\mu \chi_a(x)) T_a + g A_{\mu a} \chi_b(x) f^{abc} T_c$$
$$= -(\partial_\mu \chi_a(x)) T_a + g A_{\mu c} \chi_b(x) f^{cba} T_a$$
$$= -(\partial_\mu \chi_a(x)) T_a - g f^{abc} \chi_b(x) A_{\mu c} T_a$$

$$\rightsquigarrow \delta A_{\mu a} = -(\partial_\mu \chi_a(x)) - g f_{abc} \chi_b(x) A_{\mu c}$$

formal way or enough to factor out → formal way probably better?!

$$\text{as } \delta A_{\mu a} \underbrace{tr(T_k T_a)}_{1/2 \delta_{ka}} = -(\partial_\mu \chi_a(x)) \underbrace{tr(T_k T_a)}_{1/2 \delta_{ka}} - g f^{abc} \chi_b(x) A_{\mu c} \underbrace{tr(T_k T_a)}_{1/2 \delta_{ka}}$$

b) $\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x)$

$$\mathcal{L} \longmapsto \bar{\psi}'(x)(i\gamma^\mu D'_\mu - m)\psi'(x) = \bar{\psi}'(x) i\gamma^\mu D'_\mu \psi'(x) - m\bar{\psi}'(x)\psi'(x)$$

$$= \bar{\psi}(x) U^\dagger(x) i\gamma^\mu \underbrace{D'_\mu \psi'(x)}_{U(x) D_\mu \psi(x),\ \text{see a)}} - m\bar{\psi}(x) U^\dagger(x) U(x)\psi(x)$$

$$= \bar{\psi}(x) U^\dagger(x) U(x) i\gamma^\mu D_\mu \psi(x) - m\bar{\psi}(x) U^\dagger(x) U(x)\psi(x)$$

$$\overset{U^\dagger U = 1}{=} \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) \quad \text{invariant}$$

c) $ig(F^a_{\mu\nu} T^a)\psi := (D_\mu D_\nu - D_\nu D_\mu)\psi$

$$= \{(\partial_\mu + igA^a_\mu T^a)(\partial_\nu + igA^b_\nu T^b) - (\partial_\nu + igA^b_\nu T^b)(\partial_\mu + igA^a_\mu T^a)\}\psi$$

$$= \{\underline{\partial_\mu\partial_\nu} + ig\,\partial_\mu A^b_\nu T^b + igA^a_\mu T^a \partial_\nu - g^2 A^a_\mu T^a A^b_\nu T^b$$
$$\underline{-\partial_\nu\partial_\mu} - ig\,\partial_\nu A^a_\mu T^a - igA^b_\nu T^b \partial_\mu + g^2 A^b_\nu T^b A^a_\mu T^a\}\psi$$

$$= \{ig(\partial_\mu A^b_\nu)T^b - ig(\partial_\nu A^a_\mu)T^a + g^2 A^b_\nu T^b A^a_\mu T^a - g^2 A^a_\mu T^a A^b_\nu T^b\}\psi$$

$$= ig\{(\partial_\mu A^a_\nu)T^a - (\partial_\nu A^a_\mu)T^a + \frac{g}{i} A^b_\nu A^a_\mu (if^{bac} T^c)\}\psi$$

$$= ig\{(\partial_\mu A^a_\nu) - (\partial_\nu A^a_\mu) + g A^c_\nu A^b_\mu f^{cba}\} T^a \psi$$

$$\rightsquigarrow F^a_{\mu\nu} = (\partial_\mu A^a_\nu) - (\partial_\nu A^a_\mu) - g f^{abc} A^b_\mu A^c_\nu$$

d) Had $D_\mu' U(x) = U(x) D_\mu$

$$\rightsquigarrow [(D_\mu D_\nu - D_\nu D_\mu)\psi]' = (D_\mu' D_\nu' - D_\nu' D_\mu')\psi'$$
$$= (D_\mu' D_\nu' - D_\nu' D_\mu') U(x)\psi = U(x)(D_\mu D_\nu - D_\nu D_\mu)\psi$$

Why not use what we found for $F_{\mu\nu}^a$?

thus $F_{\mu\nu} := F_{\mu\nu}^a T^a \longmapsto F'_{\mu\nu}$ i.e. $F_{\mu\nu}\psi \longmapsto F'_{\mu\nu}\psi'$

$$F_{\mu\nu}\psi = F_{\mu\nu}^a T^a \psi = \frac{1}{ig}(D_\mu D_\nu - D_\nu D_\mu)\psi$$
$$\longmapsto \frac{1}{ig}[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = \frac{1}{ig} U(x)(D_\mu D_\nu - D_\nu D_\mu)\psi$$

What for the „hint"? Could do it w/o applying to wave function?

$$= U(x) F_{\mu\nu}^a \underset{U^\dagger U}{\hat{T}^a}\psi = U(x) F_{\mu\nu} U^\dagger U \psi = \underbrace{U(x) F_{\mu\nu} U^{-1}}_{F'_{\mu\nu}}\psi'$$

For an infinitesimal trafo, it follows,

$$F'_{\mu\nu} = U(x) F_{\mu\nu} U^{-1}(x) = (1 + ig\chi^a(x)T^a + \mathcal{O}(\chi_a^2)) F_{\mu\nu} (1 - ig\chi^b(x)T^b + \mathcal{O}(\chi_b^2))$$

$$\Leftrightarrow F_{\mu\nu}^{a\prime} T^a = (1 + ig\chi^a(x)T^a + \mathcal{O}(\chi_a^2)) F_{\mu\nu}^c T^c (1 - ig\chi^b(x)T^b + \mathcal{O}(\chi_b^2))$$

$$= F_{\mu\nu}^c T^c + ig\chi^a(x)T^a F_{\mu\nu}^c T^c - ig F_{\mu\nu}^c T^c \chi^b(x) T^b + \mathcal{O}(\chi_a\chi_b) + \mathcal{O}(\chi_a^2) + \mathcal{O}(\chi_b^2)$$

$$= F_{\mu\nu}^a T^a + ig\chi^a(x) F_{\mu\nu}^c (T^a T^c - T^c T^a)$$

$$= F_{\mu\nu}^a T^a + ig\chi^a(x) F_{\mu\nu}^c (i f^{acb} T^b)$$

$$= F_{\mu\nu}^a T^a - g\chi^b(x) F_{\mu\nu}^c f^{bca} T^a$$

$$= \underbrace{\left(F_{\mu\nu}^a - g f^{abc}\chi^b(x) F_{\mu\nu}^c\right)}_{F_{\mu\nu}^{a\prime}} T^a$$

✓

„The trace is taken over the matrix entries of the generators"? ⇝ Just introduce to next part w. $\mathrm{tr}(T^a T^b)$ is what remains

e)

$$\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \longmapsto \mathrm{tr}(U F_{\mu\nu} U^{-1} U F^{\mu\nu} U^{-1})$$

$$= \mathrm{tr}(U F_{\mu\nu} F^{\mu\nu} U^{-1}) \underset{\text{cyclic}}{\overset{\text{trace}}{=}} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \text{ invariant}$$

field strength tensor here $F_{\mu\nu}$ or $F_{\mu\nu}^a$? ⇝ $F_{\mu\nu}$ ✓

$$\rightsquigarrow \mathcal{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \tfrac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

proper units? ✓

⇝ [illegible] $\mathcal{L}$ (4 mass dim. [illegible] eq. 8 and $(\,)^2$ it)

f) Have $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

For SU(2), we have the Pauli-matrices: $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with:

$$\sigma^a \sigma^b = \delta^{ab} \mathbb{1} + i\epsilon^{abc}\sigma^c$$

Defining $T^a = \frac{1}{2}\sigma^a$ yields

$$\mathrm{tr}(T^a T^b) = \frac{1}{4}\mathrm{tr}(\sigma^a \sigma^b) = \frac{1}{4}\left\{\mathrm{tr}\underbrace{(\delta^{ab}\mathbb{1})}_{2} + i\epsilon^{abc}\underbrace{\mathrm{tr}(\sigma^c)}_{=0,\ \text{traceless}}\right\}$$

$$= \frac{1}{2}\delta^{ab} \checkmark$$

g) $$\frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{2}\mathrm{tr}(F^a_{\mu\nu}T^a F^{\mu\nu b} T^b) = \frac{1}{2}F^a_{\mu\nu}F^{\mu\nu b}\,\mathrm{tr}(T^a T^b)$$

$$= \frac{1}{4}\delta^{ab} F^a_{\mu\nu} F^{\mu\nu b} = \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}$$

1/2 in front of self coupling term?
⇝ maybe canonical conv.

✓

Where from $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$?
⇝ group theory; $\mathrm{tr}(T^a T^b)$ as a matrix, hermitian can be diagonalized
⇝ $x(1_1)$

2) $q \mapsto U(x)\, q \quad w/ \quad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

$$U(x) = e^{ig\, \chi_a(x) T_a} = (\mathbb{1} + ig\, \chi_a(x) T_a + \mathcal{O}(\chi_a^2))$$

trafo of $\epsilon_{ijk} \mapsto \det U \epsilon_{ijk}$?
$\rightsquigarrow$ Should not change the integration measure

a) $$w = \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, q_i q_j q_k \longmapsto \sum_{i,j,k,l,m,n=1}^{3} \epsilon_{ijk}\, U_{il} q_l\, U_{jm} q_m\, U_{kn} q_n$$

where we sum over indices appearing twice

Why only under infinitesimal?
$\rightsquigarrow$ did the general case in the tutorial

$$\rightsquigarrow w \mapsto \sum_{l,m,n=1}^{3} (\det U)\, \epsilon_{lmn}\, q_l q_m q_n = w$$

Or for infinitesimal SU(3) trafos:

$$w \mapsto \sum_{i,j,k=1}^{3} \epsilon_{ijk} \left(\mathbb{1}_{il} + ig\, \chi_a(x) (T_a)_{il} + \mathcal{O}(\chi_a^2)\right) q_l$$
$$\times \left(\mathbb{1}_{jm} + ig\, \chi_b(x) (T_b)_{jm} + \mathcal{O}(\chi_b^2)\right) q_m$$
$$\times \left(\mathbb{1}_{kn} + ig\, \chi_c(x) (T_c)_{kn} + \mathcal{O}(\chi_c^2)\right) q_n$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, q_i q_j q_k + ig \sum_{i,j,k=1}^{3} \epsilon_{ijk} \Big\{ \chi_a(x) (T_a)_{il}\, \mathbb{1}_{jm} \mathbb{1}_{kn}$$
$$+ \chi_b(x) (T_b)_{jm}\, \mathbb{1}_{il} \mathbb{1}_{kn}$$
$$+ \chi_c(x) (T_c)_{kn}\, \mathbb{1}_{il} \mathbb{1}_{jm} \Big\}\, q_l q_m q_n$$

antisym. under exchange of i,j,k

+ "HO" symm. under $i \leftrightarrow j \leftrightarrow k$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, q_i q_j q_k = w$$

b) $$m = \sum_{i=1}^{3} \bar{q}_i q_i \longmapsto \sum_{i=1}^{3} \left(\overline{U_{im} q_m}\right) (U_{in} q_n)$$
$$= \sum_{i=1}^{3} \bar{q}_m (U_{im})^{\dagger} U_{in} q_n$$

$$\rightsquigarrow m \mapsto \sum_{i=1}^{3} \bar{q}_m (U^{\dagger})_{mi} U_{in} q_n = \sum_{i=1}^{3} \bar{q}_i q_i$$

or infinitesimal:

$$m \to \sum_{i=1}^{3} \bar{q}_m \left(\mathbb{1}_{im} - ig\, \chi_a(x) (T_a)^{\dagger}_{im} + \mathcal{O}(\chi_a^2)\right) \left(\mathbb{1}_{in} + ig\, \chi_b(x) (T_b)_{in} + \mathcal{O}(\chi_b^2)\right) q_n$$
$$= \sum_{i=1}^{3} \bar{q}_i q_i + \sum_{i=1}^{3} \bar{q}_m \left(ig\, \chi_a(x) (T_a)_{in} \mathbb{1}_{im} - ig\, \chi_a(x) (T_a)^{\dagger}_{im} \mathbb{1}_{in}\right) + \text{HO}$$

$= m + \bar{q}_m \left(ig\, \lambda_a(x) (T_a)_{mn} - ig\, \lambda_a(x) (T_a^\dagger)_{mn} \right) q_n$

$= m$ as T_a hermitian

c) For $\bar{q}_1 q_1 \longmapsto (\overline{U_{1i} q_i})(U_{1j} q_j) = \bar{q}_i (U^\dagger)_{i1} U_{1j} q_j$

$\neq \bar{q}_1 q_1$ in general

$q_1 \longrightarrow q_2$ $\qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\bar{q}_1 q_1 \longrightarrow \bar{q}_2 q_2$ $\qquad$ Counterexample