Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
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I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.

19.12.2017 Theoretical Particle Physics Exercise 11

Marvin Zanke

1) a) $$[T_1, T_2] = \frac{1}{4}\left\{\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= iT_3$$

✓ Which generator commutes w/ these 3? ⇝ T_8

$$[T_1, T_3] = \frac{1}{4}\left\{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -iT_2$$

✓ Anything else for SU(2) algebra? ⇝ No, it's fine like this in total.

$$[T_2, T_3] = \frac{1}{4}\left\{\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= iT_1$$

$$\left[\sum_a T^a T^a, T^b\right] = \sum_a i \underbrace{f_{abc}}_{\text{antisym}} \underbrace{\{T_a, T_c\}}_{\text{sym}} = 0$$

⇝ $[T_i, T_j] = i\epsilon_{ijk} T_k$

b) $$T^1T^1 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^1T^2 = \frac{1}{4}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^1T^3 = \frac{1}{4}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^1T^4 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^1T^5 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix}, \quad T^1T^6 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^1T^7 = \frac{1}{4}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^1T^8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^2T^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2T^3 = \frac{1}{4}\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^2T^4 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2T^5 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^2T^6 = \frac{1}{4}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2T^7 = \frac{1}{4}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^2T^8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^3T^3 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3T^4 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^3T^5 = \frac{1}{4}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3T^6 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^3T^7 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3T^8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^4T^4 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^4T^5 = \frac{1}{4}\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$T^4T^6 = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^4T^7 = \frac{1}{4}\begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^4T^8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T^5T^5 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^5T^6 = \frac{1}{4}\begin{pmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^5T^7 = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^5T^8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & 0 & 2i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$T^6T^6 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^6T^7 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$T^6T^8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T^7T^7 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^7T^8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & i & 0 \end{pmatrix}$$

$$T^8T^8 = \frac{1}{4\cdot 3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \Longrightarrow tr(T^aT^b) = C\,\delta^{ab}, \quad C = 1/2$$

c) $$\sum T^a T^a = \frac{1}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$+ \frac{1}{4}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$+ \frac{1}{4}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$+ \frac{1}{12}\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix}$$

$$= \frac{3}{4}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$+ \frac{1}{12}\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix} = \begin{pmatrix}4/3&0&0\\0&4/3&0\\0&0&4/3\end{pmatrix}$$

✓ Why can it always be written like this? $\rightsquigarrow [T^a, T^2] = 0$ $\rightsquigarrow$ Schur's Lemma

$$= 4/3\, \mathbb{1} \quad \rightsquigarrow \quad C_2 = 4/3$$

- The dimension of the repr. is $d(r) = 3$
- the dimension of the adjoint rep. is $d(G) = 8$

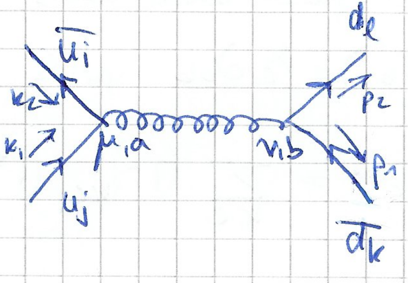
$$\rightsquigarrow \quad d(r)\, C_2 \overset{!}{=} d(G)\, C$$

✓ adjoint repr.? $\rightsquigarrow$ from Lagrangian $D_\mu \to \partial_\mu$ and want $D_\mu \psi$ to fulfill st. $\rightsquigarrow$ need $t^a = f^{abc}$ and this is the adjoint rep. for gluons then. SU(3) implies fund. rep. for the quarks

$$\Leftrightarrow \quad 3 \cdot 4/3 \overset{!}{=} 8 \cdot 1/2 \quad ✓ \text{ fulfilled}$$

$$[T_a, T_b] = i f_{abc} T_c.$$

2) $\bar{u}_i u_j \longmapsto \bar{d}_k d_l$



S-Channel

why sum over group index of gluon a?
⇝ all states are possible, take all into account.

a)

$$iM = \sum_{a,b} \bar{v}_{k_2} \left\{ -ig (T^a)_{ij} \gamma_\mu \right\} u_{k_1} \left(\frac{-ig^{\mu\nu}\delta^{ab}}{s+i\epsilon} \right)$$

$$\times\ \bar{u}_{p_2} \left\{ -ig (T^b)_{lk} \gamma_\nu \right\} v_{p_1}$$

$$= ig^2 \sum_a \left(\bar{v}_{k_2} (T^a)_{ij} \gamma_\mu u_{k_1} \right) \frac{1}{s+i\epsilon} \left(\bar{u}_{p_2} (T^a)_{lk} \gamma^\mu v_{p_1} \right)$$

„including the color factor" - how to w/o?
⇝ they just meant $(T^a)_{ij}(T^a)_{lk}$

No color indices on the spinors u_i?
⇝ not necessary, information is in $(T^a)_{ij}$, u_i's carries unnecessary info as the blue spinor is equal to red etc

b) $|M|^2 = M^* M = M^\dagger M$

$$M^\dagger = ig^2 v_{p_1}^\dagger (\gamma^0 \gamma^\mu \gamma^0)(T^a)^*_{lk} \gamma^0 u_{p_2} \frac{1}{s+i\epsilon} u_{k_1}^\dagger (\gamma^0 \gamma_\mu \gamma^0)(T^a)^*_{ij} \gamma^0 v_{k_2}$$

Sum over implied

$$= ig^2 \bar{v}_{p_1} \gamma^\mu (T^a)_{kl} u_{p_2} \frac{1}{s+i\epsilon} \bar{u}_{k_1} \gamma_\mu (T^a)_{ji} v_{k_2}$$

T hermitian

What color does a gluon carry? 2x (anti) color? $g\bar{g}$ in → $g\bar{g}$ also out? Why 8 generators then?
⇝ $3 \times \bar{3} = 8 \oplus 1$
yes if in, then the same out, gluon carries those lin. comb.

$$⇝ M^\dagger M = g^4 \bar{v}_{p_1} \gamma^\mu (T^a)_{kl} u_{p_2} \frac{1}{s+i\epsilon} \bar{u}_{k_1} \gamma_\mu (T^a)_{ji} v_{k_2}$$

Sum over a,b implied

$$\times\ \bar{v}_{k_2} (T^b)_{ij} \gamma_\nu u_{k_1} \frac{1}{s+i\epsilon} \bar{u}_{p_2} (T^b)_{lk} \gamma^\nu v_{p_1}$$

Why not use $tr(T^a T^b)$ in d) again?
⇝ just used again in tutorial (e.g. prepared in specific state, can't sum but take one element.

$$\Rightarrow |M|^2 = |M_c|^2 \, |M_{ee\mu\mu}|^2 \cdot \frac{g^4}{e^4}$$

$$|M_c|^2 = (T^a)_{kl} (T^a)_{ji} (T^b)_{ij} (T^b)_{lk}$$

$$= (T^a)_{kl} (T^b)_{lk} \ (T^a)_{ji} (T^b)_{ij}$$

c) Summing over the final state colors (indices k, ℓ) and using $\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$ yields

$$|M_c'|^2 = \sum_{k,\ell} |M_c|^2 = \sum_{k,\ell} (T^a)_{k\ell} (T^b)_{\ell k} (T^a)_{ji} (T^b)_{ij}$$

$$= \mathrm{tr}(T^a T^b) (T^a)_{ji} (T^b)_{ij} = \frac{1}{2} \delta^{ab} (T^a)_{ji} (T^b)_{ij} = \frac{1}{2} (T^a)_{ji} (T^a)_{ij}$$

d) We want to calculate $\frac{1}{9} \sum_{i,j=1}^{3} \underbrace{(T^a)_{ji} (T^a)_{ij}}_{(*) + \text{sum conv.}}$ explicitly

$$(*) = \underbrace{(T^a)_{11} (T^a)_{11}}_{\neq 0 \text{ for } T^3, T^8} + \underbrace{(T^a)_{12} (T^a)_{21}}_{T^1, T^2} + \underbrace{(T^a)_{13} (T^a)_{31}}_{T^4, T^5}$$
$$+ \underbrace{(T^a)_{21} (T^a)_{12}}_{T^1, T^2} + \underbrace{(T^a)_{22} (T^a)_{22}}_{T^3, T^8} + \underbrace{(T^a)_{23} (T^a)_{32}}_{T^6, T^7}$$
$$+ \underbrace{(T^a)_{31} (T^a)_{13}}_{T^4, T^5} + \underbrace{(T^a)_{32} (T^a)_{23}}_{T^6, T^7} + \underbrace{(T^a)_{33} (T^a)_{33}}_{T^8}$$

$$= \frac{1}{4} + \frac{1}{12} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{12} + \frac{1}{4} + \frac{1}{4}$$
$$+ \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{3} = \frac{14}{4} + \frac{1}{6} + \frac{1}{3} = \frac{24}{6}$$

$$= 4$$

$$\rightsquigarrow |\overline{M}_c|^2 = \frac{1}{9} \sum_{i,j=1}^{3} |M_c'|^2 = \frac{4}{18} = \frac{2}{9}$$

Color factors.

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