# Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
https://www.physics-and-stuff.com/

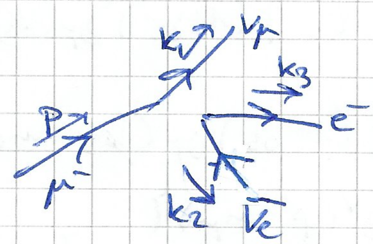
**I raise no claim to correctness and completeness of the given solutions! This equally applies to the corrections mentioned above.**

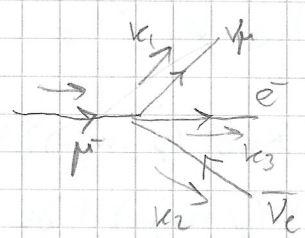
10.01.2018 **Theoretical Particle Physics Homework No. 12**

1) Consider $\mu^-(p) \to \nu_\mu(k_1)\, \bar{\nu}_e(k_2)\, e^-(k_3)$

only 1 diagram? anyways

a)




What does $m_W \gg m_\mu$ mean here? It's fixed anyways? Not equal to condition "large $s$"?

$$\frac{-ig}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)$$

$$\left(\frac{g_W}{2\sqrt{2}}\right)^2 \frac{1}{m_W^2}$$

$$\frac{g_W^2}{8m_W^2} \to \frac{G_F}{\sqrt{2}}$$

$$iM = \frac{iG_F}{\sqrt{2}}\, \bar{u}^s_{\nu_\mu}(k_1)\gamma^\mu(1-\gamma_5)u^r_\mu(p) \times \bar{u}^m_{e^-}(k_3)\gamma_\mu(1-\gamma_5)v^n_{\nu_e}(k_2)$$

b) $\bar{\Gamma} = \gamma_0\Gamma^\dagger\gamma_0 \rightsquigarrow \overline{(\gamma_\mu\gamma_5)} = \gamma_0(\gamma_\mu\gamma_5)^\dagger\gamma_0 = \gamma_0\gamma_5^\dagger\gamma_\mu^\dagger\gamma_0$
$= \gamma_0\gamma_5\gamma_0\gamma_\mu = -\gamma_5\gamma_\mu = \gamma_\mu\gamma_5$

What for $\bar{\Gamma}$? we don't calc. $\bar{M}$ but $M^\dagger$?
$\rightsquigarrow \overline{(\bar{u}\Gamma u)} = \bar{u}\bar{\Gamma}u$
$= (\bar{u}\Gamma u)^\dagger = u^\dagger\Gamma^\dagger\gamma_0 u$
$= \bar{u}\gamma^0\Gamma^\dagger\gamma_0 u$

$$M^\dagger = \frac{G_F}{\sqrt{2}}\, u^{r\dagger}_\mu(p)(1-\gamma_5^\dagger)\gamma^{\mu\dagger}\gamma_0 u^s_{\nu_\mu}(k_1) \times v^{n\dagger}_{\nu_e}(k_2)(1-\gamma_5^\dagger)\gamma_\mu^\dagger\gamma_0 u^m_{e^-}(k_3)$$

$$= \frac{G_F}{\sqrt{2}}\, u^{r\dagger}_\mu(p)(1-\gamma_5)\gamma_0\gamma^\mu\gamma_0\gamma_0 u^s_{\nu_\mu}(k_1) \times v^{n\dagger}_{\nu_e}(k_2)(1-\gamma_5)\gamma_0\gamma_\mu\gamma_0\gamma_0 u^m_{e}(k_3)$$

$$= \frac{G_F}{\sqrt{2}}\, \bar{u}^r_\mu(p)(1+\gamma_5)\gamma^\mu u^s_{\nu_\mu}(k_1) \times \bar{v}^n_{\nu_e}(k_2)(1+\gamma_5)\gamma_\mu u^m_{e^-}(k_3)$$

Factor ½? Spin average or not because only left handed comp.? But only small in V-A th.? → if neutrinos in initial state then no averaging even if energy low; we don't know right handed neutrinos right handed leptons don't interact w/ anything (not antineutrinos included)

$$\Rightarrow \overline{|M|^2} = \frac{1}{2}\sum_{r,s,n,m} M^\dagger M$$

$$= \frac{G_F^2}{4}\sum_{r,s,n,m} \bar{u}^r_\mu(p)(1+\gamma_5)\gamma^\mu u^s_{\nu_\mu}(k_1)\bar{v}^n_{\nu_e}(k_2)(1+\gamma_5)\gamma_\mu u^m_{e^-}(k_3) \times \bar{u}^s_{\nu_\mu}(k_1)\gamma^\nu(1-\gamma_5)u^r_\mu(p)\bar{u}^m_{e^-}(k_3)\gamma_\nu(1-\gamma_5)v^n_{\nu_e}(k_2)$$

$$= \frac{G_F^2}{4}\sum_{r,s,n,m} \mathrm{tr}\left\{\bar{u}_\mu(p)(1+\gamma_5)\gamma^\mu u^s_{\nu_\mu}(k_1)\bar{u}^s_{\nu_\mu}(k_1)\gamma^\nu(1-\gamma_5)u^r_\mu(p)\right\} \times \mathrm{tr}\left\{\bar{v}^n_{\nu_e}(k_2)(1+\gamma_5)\gamma_\mu u^m_{e^-}(k_3)\bar{u}^m_{e^-}(k_3)\gamma_\nu(1-\gamma_5)v^n_{\nu_e}(k_2)\right\}$$

$$= \frac{G_F^2}{4} \, tr\left\{(1+\gamma_5)\gamma^\mu (\not{k}_1 + m_{\nu_\mu})\gamma^\nu (1-\gamma_5)(\not{p} + m_\mu)\right\}$$
$$\times \, tr\left\{(1+\gamma_5)\gamma_\mu (\not{k}_3 + m_e)\gamma_\nu (1-\gamma_5)(\not{k}_2 + m_{\nu_e})\right\}$$

$$= \frac{G_F^2}{4} \, tr\left\{(1+\gamma_5)^2 \gamma^\mu \not{k}_1 \gamma^\nu (\not{p} + m_\mu)\right\} \times tr\left\{(1+\gamma_5)^2 \gamma_\mu (\not{k}_3 + m_e)\gamma_\nu \not{k}_2\right\}$$

$$= G_F^2 \, tr\left\{(1+\gamma_5)\gamma^\mu \not{k}_1 \gamma^\nu (\not{p} + m_\mu)\right\} \times tr\left\{(1+\gamma_5)\gamma_\mu (\not{k}_3 + m_e)\gamma_\nu \not{k}_2\right\}$$

$$= G_F^2 \, tr\left\{\gamma^\mu \not{k}_1 \gamma^\nu (\not{p} + m_\mu) + \gamma_5 \gamma^\mu \not{k}_1 \gamma^\nu (\not{p} + m_\mu)\right\}$$
$$\times \, tr\left\{\gamma_\mu (\not{k}_3 + m_e)\gamma_\nu \not{k}_2 + \gamma_5 \gamma_\mu (\not{k}_3 + m_e)\gamma_\nu \not{k}_2\right\}$$

$tr(\text{odd } \gamma\text{'s} = 0)$ & $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$

Why $tr(\gamma^\mu\gamma^\nu\gamma^5) = 0$?

$$= G_F^2 \, tr\left\{\gamma^\mu \not{k}_1 \gamma^\nu \not{p} + \gamma_5 \gamma^\mu \not{k}_1 \gamma^\nu \not{p}\right\} \times tr\left\{\gamma_\mu \not{k}_3 \gamma_\nu \not{k}_2 + \gamma_5 \gamma_\mu \not{k}_3 \gamma_\nu \not{k}_2\right\}$$

$$= G_F^2 \left\{4 k_{1\lambda} p_\sigma (\eta^{\mu\lambda}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\lambda\sigma} + \eta^{\mu\sigma}\eta^{\lambda\nu}) + 4i\, k_{1\lambda} p_\sigma \epsilon^{\mu\lambda\nu\sigma}\right\}$$
$$\times \left\{4 k_3^\rho k_2^\epsilon (\eta_{\mu\rho}\eta_{\nu\epsilon} - \eta_{\mu\nu}\eta_{\rho\epsilon} + \eta_{\mu\epsilon}\eta_{\rho\nu}) + 4i\, k_3^\rho k_2^\epsilon \epsilon_{\mu\rho\nu\epsilon}\right\}$$

$$= 16 G_F^2 \left\{k_1^\mu p^\nu - \eta^{\mu\nu}(k_1 \cdot p) + p^\mu k_1^\nu + i\, k_{1\lambda} p_\sigma \epsilon^{\mu\lambda\nu\sigma}\right\}$$
$$\times \left\{k_{3\mu} k_{2\nu} - \eta_{\mu\nu}(k_3 \cdot k_2) + k_{2\mu} k_{3\nu} + i\, k_3^\rho k_2^\epsilon \epsilon_{\mu\rho\nu\epsilon}\right\}$$

$\epsilon$-tensor parts are antisym. under $\mu \leftrightarrow \nu$ and the other parts are symmetric $\rightsquigarrow$ vanish

$$= 16 G_F^2 \left\{(k_1^\mu p^\nu - \eta^{\mu\nu}(k_1 \cdot p) + p^\mu k_1^\nu)(k_{3\mu} k_{2\nu} - \eta_{\mu\nu}(k_3 \cdot k_2) + k_{2\mu} k_{3\nu}) - k_{1\lambda} p_\sigma \epsilon^{\mu\lambda\nu\sigma} k_3^\rho k_2^\epsilon \epsilon_{\mu\rho\nu\epsilon}\right\}$$

Where from the $\epsilon$-tensor contraction rel and the tr(

$$= 16 G_F^2 \left\{(k_1 \cdot k_3)(p \cdot k_2) - (k_1 \cdot p)(k_3 \cdot k_2) + (k_1 \cdot k_2)(p \cdot k_3) - (k_3 \cdot k_2)(k_1 \cdot p)\right.$$
$$+ 4(k_1 \cdot p)(k_3 \cdot k_2) - (k_2 \cdot k_3)(k_1 \cdot p) + (p \cdot k_3)(k_1 \cdot k_2) - (k_1 \cdot p)(k_3 \cdot k_2)$$
$$\left.+ (p \cdot k_2)(k_1 \cdot k_3) + 2\, k_{1\lambda} p_\sigma k_3^\rho k_2^\epsilon (\delta^\lambda_\rho \delta^\sigma_\epsilon - \delta^\lambda_\epsilon \delta^\sigma_\rho)\right\}$$

$\rightsquigarrow \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = -\frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$

$$= 16 G_F^2 \left\{2(p \cdot k_2)(k_1 \cdot k_3) + 2(p \cdot k_3)(k_1 \cdot k_2) + 2(k_1 \cdot k_3)(p \cdot k_2) - 2(k_1 \cdot k_2)(p \cdot k_3)\right\}$$

$$= 64 G_F^2 (p \cdot k_2)(k_1 \cdot k_3)$$

In dependence of t?
Only channel avail
$\rightsquigarrow$ didn't write in dep. of t in tutorial

Twice the energy of particle i?

c) $x_i = \dfrac{2(k_i \cdot p)}{m_\mu^2}$, ratio $\dfrac{E_i}{m_\mu}$ where $k_i \cdot p = m_\mu E_i$

$$\rightsquigarrow x_i = \frac{2E_i}{m_\mu}$$

Or why cms energy? → see next page; need the 2 for $m_\mu > 2|k_1|$ in the limit

$$\sum_{i=1}^{3} x_i = \frac{2(k_1\cdot p) + 2(k_2\cdot p) + 2(k_3 \cdot p)}{m_\mu^2} = \frac{2p(k_1+k_2+k_3)}{m_\mu^2}$$

$$= \frac{2p^2}{m_\mu^2} = 2 \text{, using } p^2 = m_\mu^2$$

Rest frame of CMS for decaying particle? → yes

$$\int d\Pi_3 = \left(\int \prod_{i=1}^{3} \frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_i}\right)(2\pi)^4 \delta^{(4)}(p - \textstyle\sum_i k_i)$$

$$\overset{\text{rest-frame muon}}{=} \frac{1}{32\pi^5}\frac{1}{8}\int \frac{d^3k_1}{E_1}\frac{d^3k_2}{E_2}\frac{d^3k_3}{E_3}\,\delta(m_\mu - (E_1+E_2+E_3))\,\delta^{(3)}(-\vec{k}_1 - \vec{k}_2 - \vec{k}_3)$$

$$\overset{\substack{\delta\text{-distr.}\\ \text{and } E_i = |k_i| \\ \text{as } m^2 \approx 0}}{=} \frac{1}{256\pi^5}\int \frac{d^3k_1}{|k_1|}\frac{d^3k_2}{|k_2|}\,\delta(m_\mu - |k_1| - |k_2| - |-\vec{k}_1 - \vec{k}_2|)\frac{1}{|k_1 + k_2|}$$

What is θ of $\theta_{12}$ chosen between $k_1$ & $k_2$?

$$= \frac{1}{256\pi^5}\int d|k_1|\,|k_1|\,d\cos\theta_1\,d\varphi_1 \int d|k_2|\,|k_2|\,d\cos\theta_{112}\,d\varphi_2$$

$$\times \frac{\delta\left(m_\mu - |k_1| - |k_2| - \sqrt{|k_1|^2 + |k_2|^2 + 2|k_1||k_2|\cos\theta_{12}}\right)}{\sqrt{|k_1|^2 + |k_2|^2 + 2|k_1||k_2|\cos\theta_{12}}}$$

$$= \frac{1}{32\pi^3}\int d|k_1|\,d|k_2|\,d\cos\theta_{112}\,|k_1||k_2|\,\frac{\delta\left(m_\mu - |k_1| - |k_2| - \sqrt{|k_1|^2+|k_2|^2+2|k_1||k_2|\cos\theta_{112}}\right)}{\sqrt{|k_1|^2+|k_2|^2+2|k_1||k_2|\cos\theta_{112}}}$$

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$

$$\int f(x)\,\delta(g(x))\,dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

and $g(x) = 0$ for one $x_i$ in our case!

$$= \frac{1}{32\pi^3}\int d|k_1|\,|k_1| \int d|k_2|\,|k_2| \int d\cos\theta_{112}\,\frac{\delta(\cos\theta_{112} - \cos\theta_n)}{\sqrt{|k_1|^2+|k_2|^2+2|k_1||k_2|\cos\theta_{112}}}$$

$$\times \left|-\frac{2|k_1||k_2|}{2\sqrt{|k_1|^2+|k_2|^2+2|k_1||k_2|\cos\theta_{112}}}\right|^{-1}_{\theta_{112} = \theta_n}$$

where $\theta_n$ is the solution to $g(x) = 0$

$$= \frac{1}{32\pi^3} \int d|k_1| \int d|k_2|$$

$$x_i = \frac{2E_i}{m_\mu} = \frac{2|k_i|}{m_\mu} \rightsquigarrow \frac{d|k_i|}{dx_i} = \frac{m_\mu}{2}$$

$$= \frac{1}{32\pi^3} \frac{m_\mu^2}{4} \int dx_1 \int dx_2 = \frac{m_\mu^2}{128\pi^3} \int dx_1 dx_2 \quad , \quad x_1 = \frac{2}{m_\mu^2} k_1 \cdot \overset{(m_\mu,\vec{0})}{p} = \frac{2}{m_\mu} |k_1|$$

$m_\mu = |k_1| + |k_2| + |k_3| \rightsquigarrow |k_3| = 0 \rightsquigarrow |k_1|, |k_2|$ are maximal

$\Rightarrow m_\mu \geq 2|k_1| \Longrightarrow 0 \leq x_i \leq 1$

same argument for all $x_i$

$\geq) \; x_1 + x_2 + x_3 = 2 \quad , \quad x_1 + x_2 \geq 1 \Rightarrow 1 - x_1 \leq x_2$

d)

$$\Gamma = \frac{1}{2m_\mu}\int d\Pi_3 \overline{|M|^2}$$

$$\overline{|M|^2} = 64G_F^2(p\cdot k_2)(k_1\cdot k_3)$$

and $p=(k_1+k_2+k_3) \rightsquigarrow p\cdot k_1 = k_1k_2+k_1k_3$

$$p\cdot k_2 = k_2k_1+k_2k_3$$

$$p\cdot k_3 = k_3k_1+k_3k_2$$

$$\longrightarrow \frac{m_\mu^2 x_1}{2} = k_1k_2+k_1k_3$$

$$\frac{m_\mu^2 x_2}{2} = k_1k_2+k_2k_3$$

$$\frac{m_\mu^2(x_1-x_2)}{2} = k_1k_3-k_2k_3$$

$$\frac{m_\mu^2 x_3}{2} = k_1k_3+k_2k_3 \rightarrow \frac{m_\mu^2(x_1-x_2+x_3)}{2} = 2k_1k_3$$

$$\longrightarrow \frac{m_\mu^2(2-2x_2)}{4} = k_1k_3 \rightsquigarrow k_1k_3 = \frac{m_\mu^2(1-x_2)}{2}$$

Analogously

$$k_1k_2 = \frac{m_\mu^2(1-x_3)}{2} = \frac{m_\mu^2(x_1+x_2-1)}{2}$$

$$k_2k_3 = \frac{m_\mu^2(1-x_1)}{2}$$

$$\longrightarrow \overline{|M|^2} = 64G_F^2\frac{m_\mu^2}{2}(1-x_2)(k_1+k_2+k_3)\cdot k_2$$

$$= 64G_F^2\frac{m_\mu^2}{2}(1-x_2)(k_1k_2+k_3k_2)$$

$$= 64G_F^2\frac{m_\mu^2}{2}(1-x_2)\frac{m_\mu^2}{2}x_2$$

$$= 16G_F^2 m_\mu^4 x_2(1-x_2)$$

$$= \frac{G_F^2}{2m_\mu}\frac{m_\mu^2}{128\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2\, 16m_\mu^4 x_2(1-x_2)$$

$$= \frac{m_\mu^5 G_F^2}{16\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2\, x_2(1-x_2)$$

$$= \frac{m_\mu^5 G_F^2}{16\pi^3}\int_0^1 dx_1\left\{\frac{1}{2}(1-(1-x_1)^2)-\frac{1}{3}(1-(1-x_1)^3)\right\}$$

$$= \frac{m_\mu^5 G_F^2}{16\pi^3}\int_0^1 dx_1\left\{\frac{1}{2}(2x_1-x_1^2)-\frac{1}{3}(x_1^3+3x_1-3x_1^2)\right\}$$

$$= \frac{m_\mu^5 G_F^2}{16\pi^3} \int_0^1 dx_1 \left( \frac{1}{2} x_1^2 - \frac{1}{3} x_1^3 \right) = \frac{m_\mu^5 G_F^2}{16\pi^3} \left( \frac{1}{6} - \frac{1}{12} \right) = \frac{m_\mu^5 G_F^2}{192\pi^3}$$

$\tau = \Gamma^{-1} = \cancel{3{,}656 \cdot 10^{-25}\,s}$

$m_\mu = 0{,}105\ GeV$ $= 2{,}914 \cdot 10^{-19}\ GeV$

$G_F = 1{,}166 \cdot 10^{-5} \frac{1}{GeV^2}$ $\rightsquigarrow \tau = 3{,}43 \cdot 10^{18} \frac{1}{GeV}$

$\hat{=} 2{,}25 \cdot 10^{-6}\,s$

$197\ MeV\ fm = 1 \rightsquigarrow 1\,fm = 5{,}068 \frac{1}{GeV}$

$3 \cdot 10^8\ m/s = 1 \rightsquigarrow 3 \cdot 10^8\,m = 1\,s \rightsquigarrow 1\,m = 3{,}33 \cdot 10^{-9}\,s$

$\rightsquigarrow 1\,fm = 3{,}33 \cdot 10^{-24}\,s$

$\rightsquigarrow 5{,}068 \frac{1}{GeV} = 3{,}33 \cdot 10^{-24}\,s$

$\rightsquigarrow \tau = 2{,}2\,\mu s$

$\hbar = 1 \quad ([\hbar] = J\,s)$

$\downarrow \qquad \downarrow$

$6{,}582 \cdot 10^{-16}\ eVs \qquad eV$

$= 6{,}582 \cdot 10^{-25}\ GeVs$

$\rightsquigarrow 1\,s = 1{,}519 \cdot 10^{24} \frac{1}{GeV}$

$\rightsquigarrow \tau = 2{,}258 \cdot 10^{-6}\,s$