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25.01.2018 Theoretical Particle Physics Exercise 14 Marvin Zanke

1) $\mathcal{L}_{kin} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \phi)^\dagger (D_\mu \phi)$

Gauge the global U(1) sym.; elevating it to a local sym. "2

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad D_\mu \phi = \partial_\mu \phi + ig A_\mu \phi$

a) Consider $\mathcal{L}_\phi = (D^\mu \phi)^\dagger (D_\mu \phi)$

$\mathcal{L}_\phi = (\partial_\mu - igA_\mu)\phi^\dagger (\partial^\mu + igA^\mu)\phi$

Charge not set to 1 but g?

$= \partial_\mu \phi^\dagger \partial^\mu \phi + igA^\mu (\partial_\mu \phi^\dagger)\phi - igA_\mu \phi^\dagger (\partial^\mu \phi) + g^2 A_\mu A^\mu \phi^\dagger \phi$

$\Big| \; \phi(x) = ve^{i\beta} + \frac{1}{\sqrt{2}} \{\eta(x) + i\xi(x)\}$

Why v.e.v given but no potential?

$= \partial_\mu [ve^{-i\beta} + \frac{1}{\sqrt{2}}(\eta(x) - i\xi(x))] \partial^\mu [ve^{i\beta} + \frac{1}{\sqrt{2}}(\eta(x) + i\xi(x))]$
$+ igA^\mu \{\partial_\mu [ve^{-i\beta} + \frac{1}{\sqrt{2}}(\eta(x) - i\xi(x))][ve^{i\beta} + \frac{1}{\sqrt{2}}(\eta(x) + i\xi(x))]$
$- [ve^{-i\beta} + \frac{1}{\sqrt{2}}(\eta(x) - i\xi(x))] \partial_\mu [ve^{i\beta} + \frac{1}{\sqrt{2}}(\eta(x) + i\xi(x))]\}$
$+ g^2 A_\mu A^\mu [ve^{-i\beta} + \frac{1}{\sqrt{2}}(\eta(x) - i\xi(x))][ve^{i\beta} + \frac{1}{\sqrt{2}}(\eta(x) + i\xi(x))]$

$= \frac{1}{2} \partial_\mu (\eta(x) - i\xi(x)) \partial^\mu (\eta(x) + i\xi(x))$
$+ igA^\mu \{2i \operatorname{Im}[\frac{1}{\sqrt{2}} \partial_\mu (\eta(x) - i\xi(x))(v(\cos\beta + i\sin\beta) + \frac{1}{\sqrt{2}}(\eta(x) + i\xi(x)))]$
$+ g^2 A_\mu A^\mu \{v^2 + \frac{1}{2}(\eta^2(x) + \xi^2(x)) + 2\frac{1}{\sqrt{2}} \operatorname{Re}(\underbrace{ve^{-i\beta}}_{\cos\beta - i\sin\beta}(\eta(x) + i\xi(x))\}$

$= \frac{1}{2} \partial_\mu \eta(x) \partial^\mu \eta(x) + \frac{1}{2} \partial_\mu \xi(x) \partial^\mu \xi(x)$
$- \sqrt{2} g A^\mu \{(\partial_\mu \eta(x))(v\sin\beta + \frac{1}{\sqrt{2}}\xi(x)) - (\partial_\mu \xi(x))(v\cos\beta + \frac{1}{\sqrt{2}}\eta(x))\}$
$+ g^2 A_\mu A^\mu \{v^2 + \frac{1}{2}(\eta^2(x) + \xi^2(x)) + \sqrt{2} v (\eta(x)\cos\beta + \xi(x)\sin\beta)\}$

What is the Higgs field now? $\eta(x), \xi(x)$ or the lin. comb. of those?

$\rightsquigarrow$ Mass term bilinear in A_μ: $\sim g^2 v^2 A_\mu A^\mu \hat{=} \frac{1}{2} M_A^2 A_\mu A^\mu$

$\longrightarrow M_A^2 = 2g^2 v^2$

b) $\mathcal{L}_{2A,H} = \sqrt{2} g^2 v A_\mu A^\mu (\eta(x)\cos\beta + \xi(x)\sin\beta)$

The diagonalised matrix was given by $D = \begin{pmatrix} -2m^2 & 0 \\ 0 & 0 \end{pmatrix}$, the mass matrix by

Interaction in b) not discussed in class?

$M^2 = -2m^2 \begin{pmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{pmatrix}$ with $OM^2O^T = D$ and $O = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$

As $D v_{easy} = \lambda v_{easy} \Leftrightarrow O^T D v_{easy} = \lambda O^T v_{easy}$

$\Leftrightarrow M^2 O^T v_{easy} = \lambda O^T v_{easy} \rightsquigarrow O^T v_{easy}$ is the Eigenvector to M^2

$\rightsquigarrow h_1 = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}, \; h_2 = \begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$ with $h_i = \begin{pmatrix} \eta(x) \\ \xi(x) \end{pmatrix}$

Correct coordinate space of h_1? $\hat{=} \begin{pmatrix} \eta(x) \\ \xi(x) \end{pmatrix}$?

$\rightsquigarrow \mathcal{L}_{2AH} = \sqrt{2} g^2 v A_\mu A^\mu h_1$

$F_{\mu\nu} F^{\mu\nu}$ not needed here?

2) v.e.v. in class $\langle\phi\rangle = \begin{pmatrix}0\\v\end{pmatrix}$

Want to write it as $\langle\phi\rangle = \begin{pmatrix}v\\0\end{pmatrix}$ and show that the physics don't change

"given that we already know ... that a phase trafo doesn't matter"? Doesn't matter why? Still different!

a) they are related by an SU(2) gauge trafo

$$\phi \mapsto e^{i\sum_a \alpha_a \tau_a/2}\phi$$

It is enough for us, to consider constant α_a's, as the v.e.v. don't depend on x. Thus, taking a global phase trafo, we get

Why not just take U(1) gauge trafo to rotate to $\begin{pmatrix}v\\0\end{pmatrix}$? Also a rotation in a plane?

$$\langle\phi\rangle \longrightarrow \langle e^{i\alpha_a\tau_a/2}\phi\rangle = e^{i\alpha_a\tau_a/2}\langle\phi\rangle = e^{i\alpha_a\tau_a/2}\begin{pmatrix}0\\v\end{pmatrix}$$

which can obviously be fulfilled.

This always holds? No Commutation has to be taken care of? → Yes, just split in even/odd part

b)
$$e^{i\frac{\alpha_a}{2}\tau_a} = \cos\left(\frac{\alpha_a}{2}\tau_a\right) + i\sin\left(\frac{\alpha_a}{2}\tau_a\right)$$

$$= \sum_n (-1)^n \frac{\left(\frac{\alpha_a}{2}\tau_a\right)^{2n}}{(2n)!} + i\sum_n (-1)^n \frac{\left(\frac{\alpha_a}{2}\tau_a\right)^{2n+1}}{(2n+1)!}$$

$\tau_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\tau_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\tau_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$\Rightarrow \tau_a^2 = \mathbb{1}$

$$= \sum_n (-1)^n (\tau_a)^{2n} \frac{\left(\frac{\alpha_a}{2}\right)^{2n}}{(2n)!} + i\sum_n (-1)^n (\tau_a)^{2n+1} \frac{\left(\frac{\alpha_a}{2}\right)^{2n+1}}{(2n+1)!}$$

$$= \sum_n (-1)^n \frac{\left(\frac{\alpha_a}{2}\right)^{2n}}{(2n)!} + i\tau_a \sum_n (-1)^n \frac{\left(\frac{\alpha_a}{2}\right)^{2n+1}}{(2n+1)!}$$

$$= \cos\left(\frac{\alpha_a}{2}\right) + i\tau_a \sin\frac{\alpha_a}{2}$$

looking at $i\tau_1 \begin{pmatrix} 0 \\ v \end{pmatrix} = i \begin{pmatrix} v \\ 0 \end{pmatrix}$,

$i\tau_2 \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}$,

$i\tau_3 \begin{pmatrix} 0 \\ v \end{pmatrix} = i \begin{pmatrix} 0 \\ -v \end{pmatrix}$

we notice that choosing $\alpha_2 = \pi$, $\alpha_1 = \alpha_3 = 0$ $\rightsquigarrow$ $e^{i\frac{\alpha_a}{2}\tau_a} = i\tau_2$

$$\Longrightarrow e^{i\frac{\alpha_a}{2}\tau_a} \langle\phi\rangle = i\tau_2 \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

c) $e^{i\frac{\pi}{2}\tau_2} L_L = i\tau_2 \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$

$L_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$ $\qquad = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L = \begin{pmatrix} e^- \\ -\nu_e \end{pmatrix}_L = L'_L = \begin{pmatrix} \nu_e' \\ e^{-\prime} \end{pmatrix}_L$

$$\rightsquigarrow \nu_e' = e^-, \quad e^{-\prime} = -\nu_e$$

'quarks' same way as leptons under weak – no right-handed

d) Had $F^{\mu\nu} = \sum_a F_a^{\mu\nu} \frac{\tau_a}{2}$, where $F^{\mu\nu} \to F'^{\mu\nu} = U F^{\mu\nu} U^{-1}$

and $U = e^{i\sum_a \alpha_a \tau_a/2} = i\tau_2$ in our case

$$\rightsquigarrow F'^{\mu\nu} = U F^{\mu\nu} U^{-1} = (i\tau_2) F^{\mu\nu} (-i\tau_2)$$

$$\overset{\text{sum conv.}}{=} \tau_2 F_a^{\mu\nu} \frac{\tau_a}{2} \tau_2$$

$$\tau_2 \tau_a \tau_2 = \tau_2 (2\delta_{a2}\mathbb{1} - \tau_2\tau_a) = 2\delta_{a2}\tau_2 - \tau_a = \begin{cases} \tau_2, & a = 2 \\ -\tau_a, & a \neq 2 \end{cases}$$

$$= \frac{1}{2}\left(-F_1^{\mu\nu}\tau_1 + F_2^{\mu\nu}\tau_2 - F_3^{\mu\nu}\tau_3\right)$$

$$F_a^{\mu\nu} = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g f^{abc} W_\mu^b W_\nu^c$$

$$W_1'^\mu = -W_1^\mu, \quad W_2'^\mu = W_2^\mu, \quad W_3'^\mu = -W_3^\mu$$

obviously fulfills this relation, as f^{abc} is totally antisymmetric in its indices

why does the Higgs doublet transform the same way as the lepton doublet? Reason to introduce doublets?

Compute the gauge trafo of the gauge bosons? $\rightsquigarrow$ same way in tutorial; just wanted to calculate how the W's transform

In the basis for $W^\pm$, W_3, we have

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(W_{1\mu} \mp i W_{2\mu})$$

$$W_{3\mu} = W_{3\mu}$$

$$\rightsquigarrow W'^\pm_\mu = \frac{1}{\sqrt{2}}(W'_{1\mu} \mp i W'_{2\mu}) = \frac{1}{\sqrt{2}}(-W_{1\mu} \mp i W_{2\mu})$$

$$= -\frac{1}{\sqrt{2}}(W_{1\mu} \pm i W_{2\mu}) = -W^\mp_\mu$$

$$W'_{3\mu} = -W_{3\mu}$$

$$\rightsquigarrow W'^+_\mu = -W^-_\mu, \quad W'^-_\mu = -W^+_\mu, \quad W'_3 = W_3$$

from which Lagrangian does it come? $\rightsquigarrow$ Kin. term $SU(2)_L \times U(1)_Y$ automatically from $\gamma_\mu \mapsto D_\mu$

e) $$\mathcal{L}_{int} = \frac{g}{2} \bar{L} \sum_a W^\mu_a \tau_a \gamma_\mu \frac{1-\gamma_5}{2} L,$$

$$= \frac{g}{2}(\bar{\nu}_e, \bar{e}^-)_L \left\{ W^\mu_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + W^\mu_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + W^\mu_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\times \gamma_\mu \frac{1-\gamma_5}{2} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$$

In L no right-handed amplitude? But in L can be right-handed anti-particles? $\rightsquigarrow \bar{\psi} P_L \psi = \bar{\psi}_R \psi_L$ right-handed antip. in $\bar{L}$.

$$= \frac{g}{2}(\bar{\nu}_e, \bar{e}^-) \begin{pmatrix} W_3^\mu & W_1^\mu - iW_2^\mu \\ W_1^\mu + iW_2^\mu & -W_3^\mu \end{pmatrix} \gamma_\mu \frac{1-\gamma_5}{2} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$$

$$= \frac{g}{2}\left\{ \left(\bar{\nu}_e W_3^\mu + \bar{e}^-(W_1^\mu + iW_2^\mu),\ \bar{\nu}_e(W_1^\mu - iW_2^\mu) - \bar{e}^- W_3^\mu \right) \times \gamma_\mu \frac{1-\gamma_5}{2} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \right\}$$

What to multiply the γ-matrices with? $\rightsquigarrow$ with the spinors

$$= \frac{g}{2}\left\{ \bar{\nu}_e W_3^\mu \gamma_\mu \frac{1-\gamma_5}{2} \nu_e + \bar{e}^-(W_1^\mu + iW_2^\mu)\gamma_\mu \frac{1-\gamma_5}{2}\nu_e \right.$$

Better with W^+ and W^-

$$\left. + \bar{\nu}_e(W_1^\mu - iW_2^\mu)\gamma_\mu \frac{1-\gamma_5}{2} e^- - \bar{e}^- W_3^\mu \gamma_\mu \frac{1-\gamma_5}{2} e^- \right\}$$

Why no $U(1)_Y$ here as well?

$$\longrightarrow \frac{g}{2}\left\{ \bar{e}^-(-W_3^\mu)\gamma_\mu \frac{1-\gamma_5}{2} e^- + (-\bar{\nu}_e)(W_1^\mu + iW_2^\mu)\gamma_\mu \frac{1-\gamma_5}{2} e^- \right.$$

$$\left. + \bar{e}^-(W_1^\mu - iW_2^\mu)\gamma_\mu \frac{1-\gamma_5}{2}(-\nu_e) - (-\bar{\nu}_e)(W_3^\mu)\gamma_\mu \frac{1-\gamma_5}{2}(-\nu_e) \right\}$$

Pull W's in front? $\rightsquigarrow$ yes, just a gauge field (no matrix) $\Rightarrow$ invariant by comparison

f) Assume $\exists U$ s.t. $U\begin{pmatrix}0\\v\end{pmatrix} = \begin{pmatrix}v/2\\v/2\end{pmatrix}$

$$\leadsto (0, v)\begin{pmatrix}0\\v\end{pmatrix} = v^2 \overset{!}{=} \left(v/2 \;\; v/2\right)\begin{pmatrix}v/2\\v/2\end{pmatrix} = v^2/2$$

as unitary

Cannot be a unitary transformation, which conserves lengths.