Disclaimer

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19.10.2017 Theoretical Particle Physics Homework No. 2 Marvin Zanke

1) $x_\mu' = \Lambda^\nu_\mu x_\nu$, $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\det \Lambda = 1$

$\Lambda^0_\mu x_0 + \Lambda^1_\mu x_1 \ldots$

$v = \beta c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $c=1$

Λ Sym., but Λ^ν_λ which component? $\begin{pmatrix} \Lambda^1_1 & \ldots & \Lambda^1_3 \\ \Lambda^2_1 & & \Lambda^2_3 \\ \Lambda^3_1 & & \Lambda^3_3 \end{pmatrix}$ for $x^\nu = \Lambda^\nu_\nu x^\nu$ ✓

1. $d^3x = dx\,dy\,dz$ we use the sub-matrix, containing the transformation of the spatial components:

$$\tilde{\Lambda} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \det \tilde{\Lambda} = \gamma$$

t dependence gets lost → wrong!

$$\Longrightarrow d^3x' = (\det \tilde{\Lambda})\, dx\,dy\,dz = \gamma\, dx\,dy\,dz$$

$dx' = (\gamma dx - \gamma v dt) \rightsquigarrow dx' dy dz = (\gamma dx - \gamma v dt) dy dz$

Why not just simply calc. $\Lambda d^4x \to d^4x'$? → 2 dependencies → need matrices for trafo and thus det.

2. $d^4x = dt\,dx\,dy\,dz$

$$\rightsquigarrow d^4x' = (\det \Lambda)\, dt\,dx\,dy\,dz = dt\,dx\,dy\,dz = d^4x$$

✓

What about the integral itself? ↝ Somehow no sense; everything contained in the differentials

3. $\delta(p^2 - m^2)$: Consider $\underbrace{\int d^4p}_{L.I}\, \delta(p^2-m^2) = \underbrace{m^2}_{L.I}$

Since everything but $\delta(p^2-m^2)$ in this eq. is definitely L.I., $\delta(p^2-m^2)$ is L.I. as well.

4. Consider $\int d^4p\, \delta(\underbrace{p^2-m^2}_{E^2-\vec{p}^2-m^2 = f(E)})\, \theta(p^0)$, $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$

Do we need $\theta(p^0)$ here or just w/ neg./pos. energy sol.?

$$= \int dp^0 d^3p \frac{1}{2p^0} \delta(p^0 - \sqrt{\vec{p}^2 - m^2}) = \int \frac{d^3p}{2p^0}$$

Why θ L.I.? ↝ $E' = \gamma E - \beta\gamma p_x = \gamma(E - \beta p_x)$ and no sign change; $\theta(p)$ just gives a number (L.I.) ✓

$$\rightsquigarrow \underbrace{d^4p}_{L.I}\, \underbrace{\delta(p^2-m^2)}_{L.I}\, \underbrace{\theta(p^0)}_{L.I} = \frac{d^3p}{2p^0} \quad , p^0 = E$$

and thus $\frac{d^3p}{2p^0}$ is L.I.

2) $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 |\phi|^2 = \partial\phi^* \partial^\mu \phi - m^2 \phi\phi^*$

1. $S = \int d^4x\, \mathcal{L}$

$$0 \overset{!}{=} \delta S = \int d^4x\, \delta\mathcal{L}(\phi_1(x), \partial_\mu\phi_1(x), \phi_2(x), \partial_\mu\phi_2(x))$$

$$= \int d^4x \sum_{i=1,2} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta(\partial_\mu \phi_i) \right\} \quad \delta(\partial_\mu\phi_i) = \partial_\mu(\delta\phi_i)$$

$$= \int d^4x \sum_{i=1,2} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)} \delta\phi_i \right] - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)} \right\}$$

Surface term vanishes, as $\delta\phi_i = 0$ at boundaries

$$= \int d^4x \sum_{i=1,2} \left\{ \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\right) \delta\phi_i}_{\overset{!}{=}0\text{, E.L. eq., as this holds } \forall \delta\phi_i \text{, as d.o.f.}} \right\}$$

write as $\mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*)$ → yes, all are because formed for many variables

$\delta(\partial_\mu\phi_i) = \partial_\mu(\delta\phi_i)$ → $\delta q(t) = q'(t) - q(t)$

$\frac{d}{dt}(\delta q(t)) = \frac{d}{dt}(q'(t) - q(t)) = \dot{q}'(t) - \dot{q}(t) = \delta\dot{q}(t)$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0 \Leftrightarrow \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

$$\Leftrightarrow (\Box + m^2)\phi = 0, \quad \Box = \partial_\mu\partial^\mu$$

2. $\phi(x) \rightarrow e^{i\alpha}\phi(x)$ $\rightarrow$ $\tilde{\mathcal{L}} = \partial_\mu(e^{-i\alpha}\phi^*(x))\partial^\mu(e^{i\alpha}\phi(x)) - m^2(e^{i\alpha}\phi(x))(e^{-i\alpha}\phi^*(x))$

$\phi^*(x) \rightarrow e^{-i\alpha}\phi^*(x)$ $\quad \overset{\alpha \text{ constant}}{=} \partial_\mu\phi^*(x)\partial^\mu\phi(x) - m^2\phi(x)\phi^*(x)$

$= \mathcal{L}$

What trafo is this physically → $U(1)$ global trafo of any arbitrary phase

3. $J^\mu = \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)} (\Delta\phi)_i$, $\quad \phi(x) \rightarrow e^{i\alpha}\phi(x) = (1 + i\alpha + \mathcal{O}(\alpha^2))\phi(x)$

$\phi^*(x) \rightarrow e^{-i\alpha}\phi^*(x) = (1 - i\alpha + \mathcal{O}(\alpha^2))\phi^*(x)$

$\Longrightarrow \alpha\Delta\phi(x) = i\alpha\phi(x)$

$\alpha\Delta\phi^*(x) = -i\alpha\phi^*(x)$

factor α here? $\phi(x) \to e^{i\alpha}\phi(x) = \phi + \Delta\phi = \phi + \alpha\Delta\phi$ both possible

$$\rightarrow J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^* = (\partial^\mu\phi^*(x))(i\phi(x)) + (\partial^\mu\phi(x))(-i\phi^*(x))$$

$$= i\{(\partial^\mu\phi^*(x))\phi(x) - (\partial^\mu\phi(x))\phi^*(x)\}$$

4. $\partial_\mu J^\mu = i\partial_\mu\{(\partial^\mu\phi^*(x))\phi(x) - (\partial^\mu\phi(x))\phi^*(x)\}$

$$= i\{(\partial_\mu\partial^\mu\phi^*(x))\phi(x) + (\partial^\mu\phi^*(x))(\partial_\mu\phi(x)) - (\partial_\mu\partial^\mu\phi(x))\phi^*(x) - (\partial^\mu\phi(x))(\partial_\mu\phi^*(x))\}$$

$$\overset{\text{k.g.eq.}}{=} i\{(-m^2\phi^*(x))\phi(x) - (-m^2\phi(x))\phi^*(x)\} = 0$$

fields commute? → no fermions only bosons here, commute

$$0 = \partial_t\rho + \vec{\nabla}\vec{j} \Rightarrow \partial_t \underbrace{\int d^3x\, \rho}_{Q} = -\int \vec{\nabla}\vec{j}\, d^3x = 0$$

3) $S=\int d^4x \left(-\frac{1}{4} F_{\alpha\beta}F^{\alpha\beta}\right)$, $F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$

$$0=\partial_\mu \frac{\partial L}{\partial(\partial_\mu A_\nu)}-\underbrace{\frac{\partial L}{\partial A_\nu}}_{=0} = \partial_\mu\left(-\frac{1}{4}\right)\Big\{(\partial^\mu A^\nu-\partial^\nu A^\mu)-(\partial^\nu A^\mu-\partial^\mu A^\nu)+(\partial^\mu A^\nu-\partial^\nu A^\mu)-(\partial^\nu A^\mu-\partial^\mu A^\nu)\Big\}$$

$$=\partial_\mu(-1)\left\{\partial^\mu A^\nu-\partial^\nu A^\mu\right\}=\partial_\mu F^{\nu\mu}$$

⇝ more detail ↪

Dependence of L is $\partial_\mu A_\nu$? $L(A_\mu, \partial_\mu A_\nu)$ ⇝ yes, but no A_μ dep!

Half of MW eq.? ⇝ $E^i = F^{0i} = \partial^i A^0 - \partial^0 A^i = -\nabla_i A^0 - \partial^0 A^i$ ⇝ $\vec{E}=-\vec\nabla\phi-\frac{\partial\vec A}{\partial t}$, $\vec B = \vec\nabla\times\vec A$ already built in the stress tensor

Using $F^{0i}=-E^i=-F^{i0}$ and $F^{ij}=-\epsilon_{ijk}B^k$, we find

$$0=\partial_\mu F^{\nu\mu} \Leftrightarrow 0=\partial_\mu F^{\mu\nu}=\partial_0F^{0\nu}+\partial_iF^{i\nu}$$

⇝ $\nu=0$: $0=\partial_0F^{00}+\partial_iF^{i0}=\partial_i(+E^i)=+\vec\nabla\cdot\vec E$

$$\Leftrightarrow\ 0=\vec\nabla\cdot\vec E$$

$\nu=j$: $0=\partial_0F^{0j}+\partial_iF^{ij}=\partial_t(-E^j)+\partial_i(-\epsilon_{ijk}B^k)$

$$=-\partial_tE^j+(\vec\nabla\times\vec B)^j$$

$$\Leftrightarrow\ 0=(\vec\nabla\times\vec B)-\partial_t\vec E$$

2. $\hat{T}^{\mu\nu}=T^{\mu\nu}+\partial_\lambda K^{\lambda\mu\nu}$, $K^{\lambda\mu\nu}=F^{\mu\lambda}A^\nu$

same conserved energy? equally good momentum tensor? ⇝ just added a derivative

$$T^{\mu\nu}=\underbrace{\frac{\partial L}{\partial(\partial_\mu A_\lambda)}}_{\text{see 1)}}\partial^\nu A_\lambda-\mathcal{L}g^{\mu\nu}, \quad \mathcal{L}=-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$

$$=F^{\lambda\mu}\partial^\nu A_\lambda+\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$\rightsquigarrow \hat T^{\mu\nu}=F^{\lambda\mu}\partial^\nu A_\lambda+\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}+\partial_\lambda(F^{\mu\lambda}A^\nu)$$

$$=F^{\lambda\mu}\partial^\nu A_\lambda+\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}+(\partial_\lambda F^{\mu\lambda})A^\nu+F^{\mu\lambda}\partial_\lambda A^\nu$$

$$=F^{\lambda\mu}(\partial^\nu A_\lambda-\partial_\lambda A^\nu)+\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}+(\partial_\lambda F^{\mu\lambda})A^\nu$$

$$=F^{\lambda\mu}F^\nu{}_\lambda+\underbrace{(\partial_\lambda F^{\mu\lambda})}_{=0\text{ by e.o.m}}A^\nu+\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

$$=F^{\lambda\mu}F^\nu{}_\lambda+\frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}=F^{\lambda\mu}\eta_{\lambda\kappa}F^{\nu\kappa}+\frac{1}{4}g^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta}$$

$$=-F^\mu{}_\kappa F^{\nu\kappa}+\frac{1}{4}g^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta}=F^{\kappa\nu}F^\mu{}_\kappa+\frac{1}{4}g^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta}$$

$$=F^{\lambda\nu}F^\mu{}_\lambda+\frac{1}{4}g^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta}$$

$$=\hat T^{\nu\mu}$$

$$\epsilon \equiv \hat{T}^{00} = F^{\lambda 0} F^{0}{}_{\lambda} + \frac{1}{4} g^{00} F_{\alpha\beta} F^{\alpha\beta} = F^{j0} F^{0}{}_{j} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$$

$$= -F^{j0} F^{0j} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = E^{j} \cdot E^{j} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$$

Same ϵ for T? ⇝ yes

$$F_{\alpha\beta} F^{\alpha\beta} = F_{0\beta} F^{0\beta} + F_{i\beta} F^{i\beta} = F_{0j} F^{0j} + F_{i0} F^{i0} + F_{ij} F^{ij}$$

$$= -E^{j} \cdot E^{j} - E^{i} \cdot E^{i} + \epsilon^{ijk} B^{k} \epsilon^{ije} B^{e}$$

$$= -2|\vec{E}|^{2} + 2 \delta_{ke} B^{k} B^{e} = -2|\vec{E}|^{2} + 2|\vec{B}|^{2}$$

Only $A^{i} \hat{=} \vec{A}$, not A_i?
$A^{\mu} = (\phi, \vec{A})$ so $A^{i} \hat{=} \vec{A}$

$$= |\vec{E}|^{2} + \frac{1}{4}\left(-2|\vec{E}|^{2} + 2|\vec{B}|^{2}\right) = \frac{1}{2}\left(|\vec{E}|^{2} + |\vec{B}|^{2}\right)$$

$F_{0j} = -F^{0j}$? $= -F^{j}{}_{0} = -F^{j0}$
⇝ yes, pull the index up, - for spatial, do not change order

$$S^{i} \equiv \hat{T}^{0i} = F^{\lambda 0} F^{i}{}_{\lambda} + \frac{1}{4} \underbrace{g^{0i}}_{=0} F_{\alpha\beta} F^{\alpha\beta} = F^{\lambda 0} F^{i}{}_{\lambda} = F^{j0} F^{i}{}_{j}$$

$$= -F^{j0} F^{ij} = -E^{j}\left(-\epsilon^{ijk} B^{k}\right) = (\vec{E} \times \vec{B})^{i}$$

$$\rightsquigarrow \vec{S} = (\vec{E} \times \vec{B})$$

3) 1. $$\frac{\partial F_{\alpha\beta}}{\partial(\partial_{\mu} A_{\nu})} = \delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} - \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha}$$

indices of derivative (in denominator) up, others down.

$$\frac{\partial(F_{\alpha\beta} F^{\alpha\beta})}{\partial(\partial_{\mu} A_{\nu})} = g^{\alpha\rho} g^{\beta\sigma}\left[\left(\delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} - \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha}\right) F_{\rho\sigma} + F_{\alpha\beta}\left(\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho}\right)\right]$$

$$= \left[\left(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}\right) F_{\rho\sigma} + F_{\alpha\beta}\left(g^{\alpha\mu} g^{\beta\nu} - g^{\beta\mu} g^{\alpha\nu}\right)\right]$$

$$= F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} = 4 F^{\mu\nu}$$

$$\partial_{\alpha} \frac{\partial \mathcal{L}}{\partial(\partial_{\alpha} A_{\beta})} - \frac{\partial \mathcal{L}}{\partial A_{\beta}} = 0 \rightarrow \boxed{\partial_{\alpha} F^{\alpha\beta} = 0}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c} T_{\mu\nu}$$ Einstein eq.; $R_{\mu\nu}, g_{\mu\nu}$ sym. ⇝ $T_{\mu\nu}$ sym.

$\overset{!}{=}$ sym.

Angular momentum needs to be conserved