Disclaimer

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04.11.2017 Theoretical Particle Physics HW 4

1) b) We first denote that C, P, T acting on $\psi(\vec{x}, t)$ means to act from left and right, and that we work in Ch. repr.

Always like that?

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$P^\dagger = P = P^{-1}$?

$\underline{P}$: $\psi(t,\vec{x}) \longrightarrow \gamma^0 \psi(t,-\vec{x}) = P\psi(t,\vec{x})P$

$\bar{\psi}(t,\vec{x}) \longrightarrow P\psi^\dagger(t,\vec{x})\gamma^0 P = P\psi^\dagger(t,\vec{x})P\gamma^0 = (P\psi(t,\vec{x})P)^\dagger \gamma^0$

$= \psi^\dagger(t,-\vec{x})\gamma^0\gamma^0 = \bar{\psi}(t,-\vec{x})\gamma^0$

Also for scalars from both sides?

$\rightsquigarrow \mathcal{L}_0 \longmapsto P\mathcal{L}_0 P = P\bar{\psi}(t,\vec{x})(i\gamma^\mu\partial_\mu - m)\psi(t,\vec{x})P$

$= P\bar{\psi}(t,\vec{x})P\,(i\gamma^\mu\partial_\mu - m)\,P\psi(t,\vec{x})P$

$= \bar{\psi}(t,-\vec{x})\gamma^0\,(i\gamma^\mu\partial_\mu - m)\,\gamma^0\psi(t,-\vec{x})$

P no effect on γ^μ, ∂_μ etc.? just pull by?

$\vec{x} \mapsto -\vec{x}$ $\quad = \bar{\psi}(t,-\vec{x})\,(i\gamma^\mu\tilde{\partial}_\mu - m)\,\psi(t,-\vec{x})$, w/ $\tilde{\partial}_\mu = \begin{pmatrix}\partial_0 \\ -\partial_i\end{pmatrix}$

$\overset{\downarrow}{=} \bar{\psi}(t,\vec{x})\,(i\gamma^\mu\partial_\mu - m)\,\psi(t,\vec{x})$

$\vec{x}\mapsto -\vec{x}$ ok or rather apply P to ∂_μ as well when pulling by? leading us w/ $\psi(t,-\vec{x})$

$\underline{T}$: $\psi(t,\vec{x}) \longrightarrow (-\gamma^1\gamma^3)\psi(-t,\vec{x}) = T\psi(t,\vec{x})T$

$\bar{\psi}(t,\vec{x}) \longrightarrow T\psi^\dagger(t,\vec{x})\gamma^0 T = (T\psi(t,\vec{x})T)^\dagger(\gamma^0)^*$

$= ((-\gamma^1\gamma^3)\psi(-t,\vec{x}))^\dagger\gamma^0 = \psi^\dagger(-t,\vec{x})(-\gamma^1\gamma^3)^\dagger\gamma^0$

$= -\psi^\dagger(-t,\vec{x})\gamma^0\gamma^3\gamma^1\gamma^0\gamma^0 = -\bar{\psi}(-t,\vec{x})\gamma^3\gamma^1$

If we pulled T in front of $\psi \rightsquigarrow \psi^*$?

$\rightsquigarrow \mathcal{L}_0 \longmapsto T\mathcal{L}_0 T = T\bar{\psi}(t,\vec{x})(i\gamma^\mu\partial_\mu - m)\psi(t,\vec{x})T$

$= T\bar{\psi}(t,\vec{x})T\,(-i(\gamma^\mu)^*\partial_\mu - m)\,T\psi(t,\vec{x})T$

$(\gamma^\mu)^* = \gamma^2\gamma^\mu\gamma^2 \longrightarrow \quad = -\bar{\psi}(-t,\vec{x})\gamma^3\gamma^1\,(-i\gamma^2\gamma^\mu\gamma^2\partial_\mu - m)\,(-\gamma^1\gamma^3)\psi(-t,\vec{x})$

$(\gamma^\mu)^* = \gamma^2\gamma^\mu\gamma^2$ always?

$\gamma^2\gamma^\mu\gamma^2 = \begin{cases}\gamma^\mu, \mu\neq 2 \\ -\gamma^\mu, \mu = 2\end{cases} \longrightarrow \quad = \bar{\psi}(-t,\vec{x})\gamma^3\,(-i\gamma^\mu\tilde{\partial}_\mu - m)(\gamma^1)^2\gamma^3\psi(-t,\vec{x})$

$\equiv \tilde{\gamma}^\mu$

with $\tilde{\partial}_\mu = \begin{pmatrix}-\partial^0 \\ -\partial^1 \\ \partial^2 \\ -\partial^3\end{pmatrix}$ $\qquad (\gamma^1)^2 = -\mathbb{1}$

$= -\bar{\psi}(-t,\vec{x})(-i\gamma^\mu\,\bar{\bar{\partial}}_\mu - m)(\gamma^3)^2\psi(-t,\vec{x})$, $\quad (\gamma^3)^2 = -1$

with $\bar{\bar{\partial}}_\mu = \begin{pmatrix}\partial^0 \\ -\partial^1 \\ +\partial^2 \\ -\partial^3\end{pmatrix}$

$$= \bar\psi(-t,\vec x)\,(-i\gamma^\mu \overset{\equiv}{\partial}_\mu - m)\,\psi(-t,\vec x)\,, \quad \overset{\equiv}{\partial}_\mu = \begin{pmatrix}\partial^0 \\ -\vec\partial\end{pmatrix}$$

$t \mapsto -t$

$$\overset{*}{=} \bar\psi(t,\vec x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(t,\vec x)$$

$\underline{C}$:
$$\psi(t,\vec x) \to -i\gamma^2\psi^*(t,\vec x) = C\psi(t,\vec x)C$$
$$\bar\psi(t,\vec x) \to C\psi^\dagger(t,\vec x)\gamma^0 C = (C\psi(t,\vec x)C)^\dagger\gamma^0$$
$$= (-i\gamma^2\psi^*(t,\vec x))^\dagger\gamma^0 = i\psi^T(t,\vec x)\gamma^0\gamma^2\gamma^0\gamma^0$$
$$= i(\bar\psi(t,\vec x))^*\gamma^2$$

$$\rightsquigarrow \mathcal{L}_0 \mapsto C\mathcal{L}_0C = C\bar\psi(t,\vec x)(i\gamma^\mu\partial_\mu - m)\psi(t,\vec x)C$$
$$= C\bar\psi(t,\vec x)C(i\gamma^\mu\partial_\mu - m)C\psi(t,\vec x)C$$
$$= i(\bar\psi(t,\vec x))^*\gamma^2(i\gamma^\mu\partial_\mu - m)(-i\gamma^2\psi^*(t,\vec x))$$

$\mathcal{L}_0 = \mathcal{L}_0^*$ as scalar
$$\overset{*}{=} -i\bar\psi(t,\vec x)(\gamma^2)^*(-i(\gamma^\mu)^*\partial_\mu - m)(i(\gamma^2)^*\psi(t,\vec x))$$

$(\gamma^2)^* = -\gamma^2 \longrightarrow$
$$= \bar\psi(t,\vec x)\gamma^2(-i\gamma^2\gamma^\mu\gamma^2\partial_\mu - m)\gamma^2\psi(t,\vec x)$$
$$= \bar\psi(t,\vec x)\gamma^2(-i\tilde\gamma^\mu\partial_\mu - m)\gamma^2\psi(t,\vec x)\,, \text{ w/ } \tilde\gamma^\mu = \begin{cases}\gamma^\mu, & \mu\neq 2\\ -\gamma^\mu, & \mu = 2\end{cases}$$
$$= \bar\psi(t,\vec x)(-i\tilde\gamma^\mu\partial'_\mu - m)(\gamma^2)^2\psi(t,\vec x)\,, \quad \partial'_\mu = \begin{pmatrix}-\partial^0\\ -\partial^1\\ -\partial^2\\ -\partial^3\end{pmatrix}$$

$$= -\bar\psi(t,\vec x)(i\gamma^\mu\partial_\mu - m)\psi(t,\vec x)$$

why a minus?

c) $j^\mu = \bar\psi \gamma^\mu \psi$, $j^0 \overset{\text{charge density}}{=} \rho$, $\vec j$ 3-current

What means „expected"?

$$j^\mu \xrightarrow{P} P j^\mu P = P\bar\psi(t,\vec x)\gamma^\mu\psi(t,\vec x)P = P\bar\psi(t,\vec x)P\gamma^\mu P\psi(t,\vec x)P$$

$$= \bar\psi(t,-\vec x)\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec x) = \bar\psi(t,\vec x)(\gamma^\mu)^\dagger\psi(t,-\vec x)$$

Still $-\vec x$ in argument?

$$= \begin{cases} j^\mu \\ -j^\mu \end{cases}$$

$$j^\mu \xrightarrow{T} T j^\mu T = T\bar\psi(t,\vec x)\gamma^\mu\psi(t,\vec x)T$$

$$= T\bar\psi(t,\vec x)T(\gamma^\mu)^* T\psi(t,\vec x)T$$

$$= -\bar\psi(-t,\vec x)\gamma^3\gamma^1(\gamma^\mu)^*(-\gamma^1\gamma^3)\psi(-t,\vec x)$$

$$= \bar\psi(-t,\vec x)\gamma^3\tilde\gamma^\mu(\gamma^1)^2\gamma^3\psi(-t,\vec x), \quad \text{w/ } \tilde\gamma^\mu = \begin{pmatrix}-\gamma^0\\ \gamma^1\\ +\gamma^2\\ -\gamma^3\end{pmatrix}$$

$$= \bar\psi(-t,\vec x)\tilde{\tilde\gamma}^\mu\psi(-t,\vec x), \quad \tilde{\tilde\gamma}^\mu = \begin{pmatrix}\gamma^0\\ \gamma^1\\ -\gamma^2\\ -\gamma^3\end{pmatrix}$$

$$= \begin{cases} j^\mu \\ -j^\mu \end{cases}$$

$$j^\mu \xrightarrow{C} C j^\mu C = C\bar\psi(t,\vec x)\gamma^\mu\psi(t,\vec x)C$$

$$= C\bar\psi(t,\vec x)C\gamma^\mu C\psi(t,\vec x)C$$

$$= i(\bar\psi(t,\vec x))^*\gamma^2\gamma^\mu(-i\gamma^2\psi^*(t,\vec x))$$

transpose possible?

j^μ can be transposed for each component

$$= \psi^\dagger(t,\vec x)(\gamma^2)^T(\gamma^\mu)^T(\gamma^2)^T\bar\psi(t,\vec x)^\dagger$$

$$= -\psi^\dagger(t,\vec x)\left((\gamma^2)^\dagger\right)^*\left((\gamma^\mu)^\dagger\right)^*\left((\gamma^2)^\dagger\right)^*\left(\psi^\dagger(t,\vec x)\gamma^0\right)^\dagger$$

$$= -\psi^\dagger(t,\vec x)\gamma^0(\gamma^2)^*\gamma^0(\gamma^0\gamma^\mu\gamma^0)^*(\gamma^0\gamma^2\gamma^0)^*\gamma^0\psi(t,\vec x)$$

anticommutation = sign?

$$= -\bar\psi(t,\vec x)(\gamma^2)^*(\gamma^\mu)^*(\gamma^2)^*\psi(t,\vec x)$$

$$= -\bar\psi(t,\vec x)\gamma^\mu\psi(t,\vec x)$$

Sol. of Dirac eq. depends on represent.?

2) Using the Dirac representation, and the corresponding solution of the Dirac equation

$$u(p) = \sqrt{E+m}\begin{pmatrix}\phi \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi\end{pmatrix}$$

$$v(p') = \sqrt{E+m}\begin{pmatrix}\frac{\vec{p}'\cdot\vec{\sigma}}{E+m}\chi \\ \chi\end{pmatrix}$$

again: $N=\sqrt{E+m}$ and v w/ $\vec{p}'$?

with $\phi \hat{=} \phi_1 = \begin{pmatrix}1\\0\end{pmatrix}$ or $\phi_2 = \begin{pmatrix}0\\1\end{pmatrix}$
$\chi \hat{=} \chi_1 = \begin{pmatrix}0\\1\end{pmatrix}$ or $\chi_2 = \begin{pmatrix}1\\0\end{pmatrix}$ and $E>0$ built in

we instantly find $u(p)^\dagger = \sqrt{E+m}\left(\phi^\dagger, \phi^\dagger \frac{\vec{p}\cdot\vec{\sigma}^\dagger}{E+m}\right)$

$v(p')^\dagger = \sqrt{E+m}\left(\chi^\dagger \frac{\vec{p}'\cdot\vec{\sigma}^\dagger}{E+m}\, \chi^\dagger\right)$

$\vec{p} = \frac{1}{i}\frac{\partial}{\partial x}$

as $\vec{p}^\dagger = \vec{p}$ (hermitian). We then have $\gamma^0 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$ in Dirac Rep.

$\gamma^i = \begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}$ Pauli matrices

$\vec{\sigma}^\dagger = \vec{\sigma}$ always?

a) • $\not{p} + m = p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} + m\mathbb{1}$

$$= \begin{pmatrix}E & 0\\ 0 & -E\end{pmatrix} + \vec{p}\begin{pmatrix}0 & -\vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix} + \begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}$$

$$= \begin{pmatrix}E+m & -\vec{p}\cdot\vec{\sigma}\\ \vec{p}\cdot\vec{\sigma} & -E+m\end{pmatrix}$$

• $\sum_{s=1}^{2} u(p,s)\bar{u}(p,s) = u(p,s=1)\bar{u}(p,s=1) + u(p,s=2)\bar{u}(p,s=2)$

$$= (E+m)\begin{pmatrix}\phi_1 \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi_1\end{pmatrix}\left(\phi_1^\dagger, \phi_1^\dagger\frac{\vec{p}\cdot\vec{\sigma}^\dagger}{E+m}\right)\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$$

$$+ (E+m)\begin{pmatrix}\phi_2 \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi_2\end{pmatrix}\left(\phi_2^\dagger, \phi_2^\dagger\frac{\vec{p}\cdot\vec{\sigma}^\dagger}{E+m}\right)\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$$

$$\overset{\vec{\sigma}^\dagger=\vec{\sigma}}{=} (E+m)\begin{pmatrix}\phi_1\phi_1^\dagger & -\phi_1\phi_1^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi_1\phi_1^\dagger & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi_1\phi_1^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\end{pmatrix}$$

$$+ (E+m)\begin{pmatrix}\phi_2\phi_2^\dagger & -\phi_2\phi_2^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi_2\phi_2^\dagger & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\phi_2\phi_2^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\end{pmatrix}$$

$$\phi_1\phi_1^\dagger + \phi_2\phi_2^\dagger = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \mathbb{1} \qquad (\vec{p}\cdot\vec{\sigma})^2 = p_i\sigma_i p_j\sigma_j = \frac{1}{2}p_ip_j\{\sigma_i,\sigma_j\} = \vec{p}^{\,2}$$

$$= (E+m)\begin{pmatrix}\mathbb{1} & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\frac{(\vec{p}\cdot\vec{\sigma})^2}{(E+m)^2}\end{pmatrix} = \begin{pmatrix}E+m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -\frac{\vec{p}^{\,2}}{E+m}\end{pmatrix}$$

$-\vec{p}^{\,2} = -(E^2-m^2) = -(E-m)(E+m)$

$$= \begin{pmatrix}E+m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E+m\end{pmatrix} = \not{p}+m$$

No $\not{p}$ but $\not{p}$ here in $\not{p}-m$? And even worse: only 3-component with a minus?

b) • $\not{p}-m = p^0\gamma^0 - \vec{p}\cdot\vec{\gamma} - m\mathbb{1} = \begin{pmatrix}E-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E-m\end{pmatrix}$

• $\sum_{s=1}^{2} v(p,s)\bar{v}(p,s) = v(p,s=1)\bar{v}(p,s=1) + v(p,s=2)\bar{v}(p,s=2)$

$\vec{\sigma}^\dagger = \vec{\sigma}$

$$= (E+m)\begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_1 \\ \chi_1\end{pmatrix}\left(\chi_1^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m}, \chi_1^\dagger\right)\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix}$$

$$+ (E+m)\begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_2 \\ \chi_2\end{pmatrix}\left(\chi_2^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m}, \chi_2^\dagger\right)\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix}$$

$$= (E+m)\begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_1\chi_1^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_1\chi_1^\dagger \\ \chi_1\chi_1^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\chi_1\chi_1^\dagger\end{pmatrix}$$

$$+ (E+m)\begin{pmatrix}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_2\chi_2^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\chi_2\chi_2^\dagger \\ \chi_2\chi_2^\dagger\frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\chi_2\chi_2^\dagger\end{pmatrix}$$

$$= E+m\begin{pmatrix}\frac{(\vec{p}\cdot\vec{\sigma})^2}{(E+m)^2} & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\mathbb{1}\end{pmatrix} = \begin{pmatrix}\frac{\vec{p}^{\,2}}{E+m} & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -(E+m)\end{pmatrix}$$

$$= \begin{pmatrix}E-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E-m\end{pmatrix} = \not{p}-m$$

Complex or real scalar field?

3) $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$, $\mathcal{L}_{free} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\phi^2}{2}\phi^2$

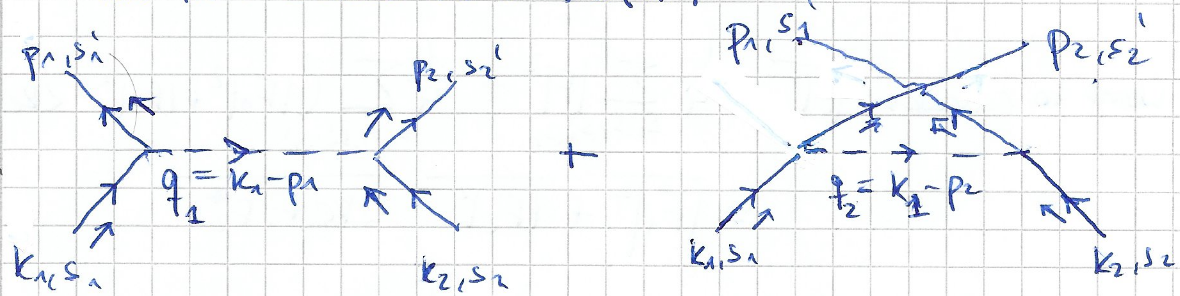
$$\mathcal{L}_{int} = -\kappa\phi\bar{\psi}\psi$$

Non trivial order $\mathcal{O}(\kappa^2)$?

$2 \to 2$ scattering of fermions: $\psi(\vec{k}_1, s_1) + \psi(\vec{k}_2, s_2) \to \psi(\vec{p}_1, s_1') + \psi(\vec{p}_2, s_2')$

a) We have 2 possible Feynman diagrams contributing to the lowest non-trivial order in P.T. $\mathcal{O}(\kappa^2)$:

Should be 4-momenta as arguments for ψ, u, v etc.?



where we get an additional "$-$" sign between those, which can be seen from the Wick-contractions and anticommutations in order to untangle:

$$A_1 \sim \langle p_1 p_2 | \phi_x \bar{\psi}_x \psi_x \phi_y \bar{\psi}_y \psi_y | k_1 k_2 \rangle$$
$$\sim \langle 0 | a_2 a_1 \phi_x \bar{\psi}_x \psi_x \phi_y \bar{\psi}_y \psi_y a_1^\dagger a_2^\dagger | 0 \rangle$$

$\rightsquigarrow$ "+" sign (2 swappings) (factor 2 cancels w/ $\frac{1}{2}$ from exponential $1/2!$)

$$A_2 \sim \langle p_1 p_2 | \phi_x \bar{\psi}_x \psi_x \phi_y \bar{\psi}_y \psi_y | k_1 k_2 \rangle$$
$$\sim \langle 0 | a_2 a_1 \phi_x \bar{\psi}_x \psi_x \phi_y \bar{\psi}_y \psi_y a_1^\dagger a_2^\dagger | 0 \rangle$$

$\rightsquigarrow$ "$-$" sign (1 swapping)

b)

$\rightsquigarrow F = F_1 - F_2 \rightsquigarrow F^* = F_1^* - F_2^*$; $FF^* = (F_1 - F_2)(F_1^* - F_2^*)$

$$= |F_1|^2 + |F_2|^2 - F_1F_2^* - F_2F_1^*$$
$$= |F_1|^2 + |F_2|^2 - 2\mathrm{Re}(F_1F_2^*)$$

And we also know that $F_1^* = F_1^\dagger$, $F_2^* = F_2^\dagger$ as they are scalars. We will calculate each part separately.

$$F_1 = \bar u(p_1,s_1')(-ik)\,u(k_1,s_1)\,\frac{i}{q_1^2-m_\phi^2}\,\bar u(p_2,s_2')(-ik)\,u(k_2,s_2)$$

$$F_2 = \bar u(p_1,s_1')(-ik)\,u(k_2,s_2)\,\frac{i}{q_2^2-m_\phi^2}\,\bar u(p_2,s_2')(-ik)\,u(k_1,s_1)$$

Particles in-distinguishable?

$$F = F_1 - F_2$$

c) $\tilde F$ results from interchanging $(\vec p_1, s_1')$ and $(\vec p_2, s_2')$
this obviously yields $\tilde F_1 = F_2$ and $\tilde F_2 = F_1$ because $\bar u u$ are scalars and can be interchanged. thus $\tilde F = \tilde F_1 - \tilde F_2 = F_2 - F_1 = -F$

d) We want to calculate $\overline{|F|^2} = \frac{1}{4}\sum_{s_1,s_2,s_1',s_2'} |F|^2 = \frac{1}{4}\sum_{s_1,s_2,s_1',s_2'}\left(|F_1|^2+|F_2|^2-2\mathrm{Re}\,F_1F_2^*\right)$

$$= \overline{|F_1|^2} + \overline{|F_2|^2} - \overline{2\mathrm{Re}\,F_1F_2^*}$$

We now calculate $F_1^* = F_1^\dagger$, $F_2^* = F_2^\dagger$

$$F_1 = \frac{-ik^2}{q_1^2-m_\phi^2}\,\bar u(p_1,s_1')u(k_1,s_1)\,\bar u(p_2,s_2')u(k_2,s_2)$$

$$F_2 = \frac{-ik^2}{q_2^2-m_\phi^2}\,\bar u(p_1,s_1')u(k_2,s_2)\,\bar u(p_2,s_2')u(k_1,s_1)$$

We will use $(\bar u_1 u_2)^\dagger = (u_1^\dagger \gamma^0 u_2)^\dagger = u_2^\dagger \gamma^{0\dagger} u_1 = \bar u_2 u_1$

$$F_1^* = \frac{ik^2}{q_1^2-m_\phi^2}\,\bar u(k_1,s_1)u(p_1,s_1')\,\bar u(k_2,s_2)u(p_2,s_2')$$

$$F_2^* = \frac{ik^2}{q_2^2-m_\phi^2}\,\bar u(k_2,s_2)u(p_1,s_1')\,\bar u(k_1,s_1)u(p_2,s_2')$$

$$\Rightarrow |F_1|^2 = \frac{k^4}{(q_1^2-m_\phi^2)^2}\,\bar u(p_1,s_1')u(k_1,s_1)\,\bar u(p_2,s_2')u(k_2,s_2)\,\bar u(k_1,s_1)u(p_1,s_1')\,\bar u(k_2,s_2)u(p_2,s_2')$$

$$\Rightarrow \overline{|F_1|^2} = \frac{k^4}{4(q_1^2-m_\phi^2)^2}\sum_{s_1,s_2,s_1',s_2'} \mathrm{tr}\left(\bar u(p_1,s_1')u(k_1,s_1)\,\bar u(k_1,s_1)u(p_1,s_1')\right)$$
$$\times\,\mathrm{tr}\left(\bar u(p_2,s_2')u(k_2,s_2)\,\bar u(k_2,s_2)u(p_2,s_2')\right)$$

$c = \mathrm{tr}(c)$

trace cyclic

$$= \frac{k^4}{4(q_1^2-m_\phi^2)^2}\left\{\mathrm{tr}\left((\not k_1+m_\psi)(\not p_1+m_\psi)\right)\mathrm{tr}\left((\not k_2+m_\psi)(\not p_2+m_\psi)\right)\right\}$$

odd γ's in trace vanishing

$$= \frac{k^4}{4(q_1^2-m_\phi^2)^2}\left\{\mathrm{tr}(\not k_1\not p_1)+\mathrm{tr}(m_\psi^2)\right\}\left\{\mathrm{tr}(\not k_2\not p_2)+\mathrm{tr}(m_\psi^2)\right\}$$

$$= \frac{k^4}{4(q_1^2-m_\phi^2)^2}\left\{4k_1\cdot p_1 + 4m_\chi^2\right\}\left\{4k_2\cdot p_2 + 4m_\chi^2\right\}$$

$$= \frac{4k^4}{(q_1^2-m_\phi^2)^2}\left\{k_1\cdot p_1 + m_\chi^2\right\}\left\{k_2\cdot p_2 + m_\chi^2\right\}$$

Can this really always be done so easily?

$$\overline{|F_2|^2} = \frac{4k^2}{(q_2^2-m_\phi^2)^2}\left\{k_1\cdot p_2 + m_\chi^2\right\}\left\{k_2\cdot p_1 + m_\chi^2\right\}$$

as F_2 results from F_1 by $p_1 \leftrightarrow p_2$ and $s_1' \leftrightarrow s_2'$

express $k_1 p_2$ etc with u, t etc.? Or angular dep.?

We now calculate the last contribution:

$$-2\,\mathrm{Re}\,F_1F_2^* = -2\,\mathrm{Re}\left\{\frac{k^4}{(q_1^2-m_\phi^2)(q_2^2-m_\phi^2)}\bar{u}(p_1,s_1')u(k_1,s_1)\bar{u}(p_2,s_2')u(k_2,s_2) \times \bar{u}(k_2,s_2)u(p_1,s_1')\bar{u}(k_1,s_1)u(p_2,s_2')\right\}$$

$$\leadsto -\overline{2\,\mathrm{Re}\,F_1F_2^*} = -\frac{k^4}{2(q_1^2-m_\phi^2)(q_2^2-m_\phi^2)}\sum_{s_1,s_1',s_2,s_2'}\mathrm{Re}\left\{\mathrm{tr}\left(\bar{u}(k_1,s_1)u(p_2,s_2')\bar{u}(p_2,s_2')u(k_2,s_2) \times \bar{u}(k_2,s_2)u(p_1,s_1')\bar{u}(p_1,s_1')u(k_1,s_1)\right)\right\}$$

$$= -\frac{k^4}{2(q_1^2-m_\phi^2)(q_2^2-m_\phi^2)}\mathrm{Re}\left\{\mathrm{tr}\left((\not{p}_2+m_\chi)(\not{k}_2+m_\chi)(\not{p}_1+m_\chi)(\not{k}_1+m_\chi)\right)\right\}$$

$$= -\frac{k^4}{2(q_1^2-m_\phi^2)(q_2^2-m_\phi^2)}\mathrm{Re}\left\{\mathrm{tr}(\not{p}_2\not{k}_2\not{p}_1\not{k}_1) + m_\chi^2\big(\mathrm{tr}(\not{p}_2\not{k}_2) + \mathrm{tr}(\not{p}_2\not{p}_1) + \mathrm{tr}(\not{p}_2\not{k}_1) + \mathrm{tr}(\not{k}_2\not{p}_1) + \mathrm{tr}(\not{k}_2\not{k}_1) + \mathrm{tr}(\not{p}_1\not{k}_1)\big) + \mathrm{tr}(m_\chi^4)\right\}$$

$$= -\frac{k^4}{2(q_1^2-m_\phi^2)(q_2^2-m_\phi^2)}\mathrm{Re}\Big\{4m_\chi^4 + 4m_\chi^2\big(p_2\cdot k_2 + p_2\cdot p_1 + p_2\cdot k_1 + k_2\cdot p_1 + k_2\cdot k_1 + p_1\cdot k_1\big) + 4\big((p_2\cdot k_2)(p_1\cdot k_1) + (p_2\cdot k_1)(k_2\cdot p_1) - (p_2\cdot p_1)(k_2\cdot k_1)\big)\Big\}$$

$$= -\frac{2k^4}{(q_1^2-m_\phi^2)(q_2^2-m_\phi^2)}\Big\{m_\chi^4 + \big(p_1\cdot k_1 + p_1\cdot k_2 + p_2\cdot k_1 + p_2\cdot k_2 + k_1\cdot k_2 + p_1\cdot p_2\big)m_\chi^2 + \big((k_1\cdot p_1)(k_2\cdot p_2) + (k_1\cdot p_2)(k_2\cdot p_1) - (p_1\cdot p_2)(k_1\cdot k_2)\big)\Big\}$$