Disclaimer

The solution at hand was written in the course of the respective class at the University of Bonn. If not stated differently on top of the first page or the following website, the solution was prepared and handed in solely by me, Marvin Zanke. Anything in a different color than the ball pen blue is usually a correction that I or a tutor made. For more information and all my material, check:
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16.11.2017 Theoretical Particle Physics Ex. 6 Marvin Zanke

1)

a) $\mathcal{L}_{tot} = \mathcal{L}_{QED} + \mathcal{L}_{\xi}$

$$= -\frac{1}{4} \underbrace{F_{\mu\nu}}_{(\partial_\mu A_\nu - \partial_\nu A_\mu)} F^{\mu\nu} + \sum_i \bar\psi_i (i\not\partial - m_i - q_i \not A)\psi_i - \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$$

Why can we add this term if not Lorentz gauge? See derivation of inverse → we can add it anyways and then see (in this exercise) what happens/changes.

E.L. eq.: $\partial_\kappa \frac{\partial \mathcal{L}}{\partial(\partial_\kappa A_\lambda)} - \frac{\partial \mathcal{L}}{\partial A_\lambda} = 0$

$$\Leftrightarrow \partial_\kappa \left\{ -\frac{1}{4}\left(F^{\kappa\lambda} - F^{\lambda\kappa} + F^{\kappa\lambda} - F^{\lambda\kappa}\right) - \frac{1}{2\xi}\left(\eta^{\mu\sigma}\delta_\mu^\kappa \delta_\sigma^\lambda (\partial_\nu A^\nu)\cdot 2\right)\right\}$$

$$-(-q_i \bar\psi_i \gamma^\lambda \psi_i) = 0 \quad \text{w/ Einstein sum convention}$$

$$\Leftrightarrow -\partial_\kappa F^{\kappa\lambda} - \partial_\kappa\left(\frac{1}{\xi}\eta^{\kappa\lambda}(\partial_\nu A^\nu)\right) \overbrace{+ q_i \bar\psi_i \gamma^\lambda \psi_i}^{\equiv J^\lambda_Q} = 0$$

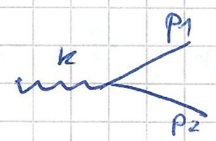
$$\Leftrightarrow \partial_\kappa F^{\kappa\lambda} = J^\lambda_Q - \frac{1}{\xi}\partial^\lambda(\partial_\nu A^\nu)$$

$$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\Leftrightarrow \partial^\mu F_{\mu\nu} = J^Q_\nu - \frac{1}{\xi}\partial_\nu(\partial_\kappa A^\kappa)$$

$$\Leftrightarrow \left(g^{\mu\nu}\partial_\lambda\partial^\lambda - (1-\frac{1}{\xi})\partial^\nu\partial^\mu\right)A_\mu = J^\nu$$

Iterated first order formalism?



b)

$$\Leftrightarrow \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{\xi}\partial_\nu(\partial_\kappa A^\kappa) = \tilde J^Q_\nu e^{-ix(p_1+p_2)}$$

Ansatz: $A_\nu(x) = f_\nu(k) e^{-ikx}$

$$\rightsquigarrow (-k^2 f_\nu(k) + k^\mu k_\nu f_\mu(k)) e^{-ikx} - \frac{1}{\xi} k_\nu k_\mu f^\mu(k) e^{-ikx} = \tilde J^Q_\nu e^{-ix(p_1+p_2)}$$

$$\Leftrightarrow \left\{(-k^2 f_\nu(k) + k_\nu(k\cdot f(k))) - \frac{1}{\xi} k_\nu (k\cdot f(k))\right\} e^{-ikx} = \tilde J^Q_\nu e^{-ix(p_1+p_2)}$$

$\rightsquigarrow$ mom. conservation: $k = (p_1+p_2)$

$$\rightsquigarrow \underbrace{\left(-k^2 g_{\mu\nu} + k_\nu k_\mu - \frac{1}{\xi} k_\nu k_\mu\right)}_{\text{needs to be inverted to find } f^\mu(k)} f^\mu(k) = \tilde J^Q_\nu$$

$$\Leftrightarrow \left(-k^2 g_{\mu\nu} + k_\nu k_\mu (1-\frac{1}{\xi})\right) f^\mu(k) = \tilde J^Q_\nu$$

$$\Leftrightarrow \left(-k^2 g_{\mu\nu} + k_\nu k_\mu \left(\frac{\xi-1}{\xi}\right)\right) f^\mu(k) = \tilde J^Q_\nu$$

$$\rightsquigarrow \left(-k^2 g_{\mu\nu} + \frac{\xi-1}{\xi} k_\nu k_\mu\right) C^{\nu\rho} \stackrel{!}{=} \delta^\rho_\mu$$

Ansatz for $C^{\nu\rho} = A(k^2) g^{\nu\rho} + B(k^2) k^\nu k^\rho$

What if Tensor not sym.? ⇝ different procedure; maybe add some term to get sym. matrix

$$\rightsquigarrow \left(-k^2 g_{\mu\nu} + \frac{\xi-1}{\xi} k_\nu k_\mu\right)\left(A(k^2) g^{\nu\rho} + B(k^2) k^\nu k^\rho\right) \stackrel{!}{=} \delta^\rho_\mu$$

$$\Longleftrightarrow -k^2 A(k^2) g^\rho_\mu - k^2 B(k^2) k_\mu k^\rho + \frac{\xi-1}{\xi} A(k^2) k^\rho k_\mu + \frac{\xi-1}{\xi} k^2 B(k^2) k_\mu k^\rho \stackrel{!}{=} \delta^\rho_\mu$$

$$\Longleftrightarrow \delta^\rho_\mu \underbrace{\left(-k^2 A(k^2)\right)}_{\stackrel{!}{=}1} + k_\mu k^\rho \underbrace{\left\{-k^2 B(k^2) + \frac{\xi-1}{\xi}\left(A(k^2) + B(k^2) k^2\right)\right\}}_{\stackrel{!}{=}0} \stackrel{!}{=} \delta^\rho_\mu$$

$$\rightsquigarrow \cdot\ A(k^2) = -\frac{1}{k^2}$$

$$\cdot\ -k^2 B(k^2) + \frac{\xi-1}{\xi}\left(-\frac{1}{k^2} + k^2 B(k^2)\right) \stackrel{!}{=} 0$$

$$\Longleftrightarrow B(k^2)\left(-k^2 + \frac{\xi-1}{\xi} k^2\right) - \frac{\xi-1}{\xi}\frac{1}{k^2} \stackrel{!}{=} 0$$

$$\Longleftrightarrow B(k^2) k^2 \left(-\frac{1}{\xi}\right) = \frac{\xi-1}{\xi}\frac{1}{k^2}$$

$$\Longleftrightarrow B(k^2) = (1-\xi)\frac{1}{k^4}$$

$$\rightsquigarrow C^{\nu\rho} = -\frac{g^{\nu\rho}}{k^2} + (1-\xi)\frac{1}{k^4} k^\nu k^\rho$$

$$= -\frac{1}{k^2}\left\{g^{\nu\rho} - (1-\xi)\frac{k^\nu k^\rho}{k^2}\right\}$$

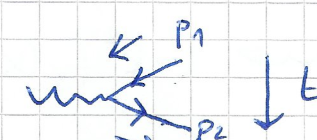
factor i? ⇝ functional derivation (QFT)

$$\rightsquigarrow D^{\mu\nu}(k) = \frac{-i}{k^2}\left\{g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right\},$$ where the i is convention

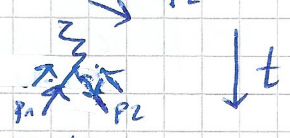
c) the amplitudes in first order are of the form

What to do for higher orders? valid?

$$iM = \frac{-i}{k^2}\left\{g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right\}\bar{u}_{p_2}\gamma_\nu u_{p_1} \times \ldots$$



or $$\frac{-i}{k^2}\left\{g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right\}\bar{u}_{p_2}\gamma_\nu v_{p_1} \times \ldots$$



We show that the extra terms don't change the amplitude:

why free eq. of motion for fermions? ⇝ no interaction

$$T_1 = \frac{i}{k^4}(1-\xi) k^\mu k^\nu \bar{u}_{p_2}\gamma_\nu u_{p_1} \underset{k=p_1-p_2}{=} \frac{i}{k^4}(1-\xi) k^\mu \bar{u}_{p_2}(\not{p}_1 - \not{p}_2) u_{p_1}$$

$$= \frac{i}{k^4}(1-\xi) k^\mu \bar{u}_{p_2}(m_{2f} - m_{2f}) u_{p_1} = 0, \text{ as } \bar{u}(\not{p}-m) = 0, (\not{p}-m)u = 0$$

$$T_2 = \frac{i}{k^4}(1-\xi) k^\mu k^\nu \bar{u}_{p_2}\gamma_\nu v_{p_1} \underset{k=p_1+p_2}{=} \frac{i}{k^4}(1-\xi) k^\mu \bar{u}_{p_2}(\not{p}_1 + \not{p}_2) v_{p_1} = 0, \text{ as } (\not{p}+m)v = 0$$

($\not{p}_2 \to m_2$, $\not{p}_1 \to -m_2$)

d) $\mathcal{L}$ still invariant under global trafo ⇝ $\psi \mapsto e^{i\alpha}\psi$ ⇝ same conserved current

$\mathcal{L}$ also inv under local trafo if $\partial^2\chi = 0$ & $A_\mu \mapsto A_\mu - \partial_\mu\chi$ (Gauge inv ⇝ conserved current)

2)

$\mathcal{L}_{free} = \partial_\mu\phi^* \partial^\mu\phi - m^2|\phi|^2$, where $|\partial_\mu\phi|^2$ is implied ! this is important for $\partial_\mu \mapsto D_\mu$ in b)

a) $\mathcal{L} \mapsto \mathcal{L}' = \partial_\mu(e^{i\alpha q}\phi^*)\partial^\mu(e^{-i\alpha q}\phi) - m^2(e^{i\alpha q}\phi)(e^{-i\alpha q}\phi^*)$

$= (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2|\phi|^2$

under $\phi \mapsto e^{i\alpha q}\phi$, $\phi^* \mapsto e^{-i\alpha q}\phi^*$

absorb i in current? How to absorb i? ⇝ $\phi + \alpha\Delta\phi$ as definition, then it works

$\rightsquigarrow J^\mu_N = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^*$

first derivative dominant on 2nd ϕ? → no

We assume $|\alpha| << 1 \rightsquigarrow \phi = (1 + i\alpha q + \mathcal{O}(\alpha^2))\phi \rightsquigarrow i\alpha\Delta\phi = i\alpha q\phi$

$\phi^* = (1 - i\alpha q + \mathcal{O}(\alpha^2))\phi^* \rightsquigarrow i\alpha\phi^* = -i\alpha q\phi$

$\rightsquigarrow J^\mu_N = \{(\partial^\mu\phi^*)\, q\phi + (\partial^\mu\phi)(-q\phi^*)\}$

$= q\{\phi(\partial^\mu\phi^*) - \phi^*(\partial^\mu\phi)\}$

prop. to prob. current?

b) $\mathcal{L}$ isn't invariant under $\phi \mapsto e^{i\alpha(x)q}\phi$, but if we replace

$\partial_\mu \mapsto D_\mu = \partial_\mu + iqA_\mu$, where

$A_\mu \mapsto A_\mu - \partial_\mu\alpha(x)$

by the covariant derivative, we find:

$\mathcal{L} = D^*_\mu\phi^* D^\mu\phi - m^2|\phi|^2 = (\partial_\mu - iqA_\mu)\phi^*(\partial^\mu + iqA^\mu)\phi - m^2|\phi|^2$,

we find:

Still $\mathcal{L}_{free}$? ⇝ no, the cov. derivative introduces the interaction.

$\mathcal{L} \mapsto \mathcal{L}' = [(\partial_\mu - iqA_\mu)e^{-iq\alpha(x)}\phi^*][(\partial^\mu + iqA^\mu)e^{iq\alpha(x)}\phi]$

$-m^2(e^{-iq\alpha(x)}\phi^*)(e^{iq\alpha(x)}\phi)$

A_μ real? No c.c. needed? ⇝ A_μ real (is a el. magn. field, no quantum field here)

$= e^{-iq\alpha(x)}[-iq\,\partial_\mu\alpha(x) - iq(A_\mu - \partial_\mu\alpha(x)) + \partial_\mu]\phi^*$

$\times e^{iq\alpha(x)}[iq\,\partial^\mu\alpha(x) + iq(A^\mu - \partial^\mu\alpha(x)) + \partial^\mu]\phi - m^2|\phi|^2$

$= [\partial_\mu - iqA_\mu]\phi^*[\partial^\mu + iqA^\mu]\phi - m^2|\phi|^2 = \mathcal{L}$

c) Looking at $\mathcal{L} = (\partial_\mu - iqA_\mu)\phi^* (\partial^\mu + iqA^\mu)\phi - m^2|\phi|^2$, we find that we have vertices $\propto \phi^*\phi A^\mu$ and $\propto \phi^*\phi A^\mu A^\nu$

we first expand $\mathcal{L}$:

Only new Vertex rules?

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) + (\partial_\mu \phi^*) iqA^\mu \phi - iqA_\mu \phi^* (\partial^\mu \phi) + q^2 A_\mu A^\mu \phi^* \phi - m^2 |\phi|^2$$

$$= (\partial_\mu \phi^*)(\partial^\mu \phi) + i A_\mu \underbrace{J^\mu_N}_{J^{old}_\mu} + \underbrace{q^2 \eta^{\mu\nu} A_\mu A_\nu \phi^* \phi}_{new} - m^2 |\phi|^2$$

$\rightsquigarrow$ new current given by sum

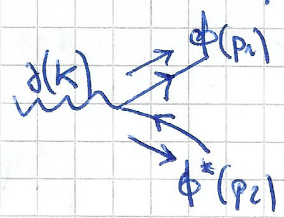
The vertex factors can be obtained from $i\frac{\partial^3 \mathcal{L}}{\partial\phi^*\partial\phi\partial A_\kappa}$ and $i\frac{\partial^4 \mathcal{L}}{\partial\phi^*\partial\phi\partial A_\kappa \partial A_\lambda}$, where we have to omit parts which do not contain all fields we are deriving for, i.e set $\phi^*, \phi, A_\nu = 0$ in the end:

Which out-going/which inc.? - ϕ, ϕ^*?

- $\phi^*\phi A_\mu$: $i\frac{\partial^3 \tilde{\mathcal{L}}}{\partial\phi^*\partial\phi\partial A_\kappa}$ with $\tilde{\mathcal{L}} = iA_\mu J^\mu_N = iqA_\mu((\partial^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi))$

Why not mix terms in derivation of Lagrangian (both vertex contributions) - $\langle \delta | \dots | \phi^*\phi \rangle = 0$ for different vertex?

$$\rightsquigarrow i\frac{\partial^2 \tilde{\mathcal{L}}}{\partial\phi^*\partial\phi}\{iq((\partial^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi))\}$$



$$= -q\frac{\partial \mathcal{L}}{\partial\phi^*}\{(\partial^\mu\phi^*) - \phi^*(-ip_1^\mu)\}$$

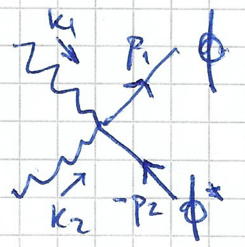
$$= -q\{ip_2^\mu + ip_1^\mu\} = -iq(p_1+p_2)^\mu$$

- $\phi^*\phi A_\mu A_\nu$: $i\frac{\partial^4 \tilde{\tilde{\mathcal{L}}}}{\partial\phi^*\partial\phi\partial A_\kappa\partial A_\lambda}$ with $\tilde{\tilde{\mathcal{L}}} = q^2\eta^{\mu\nu}A_\mu A_\nu \phi^*\phi$

$$\rightsquigarrow i\frac{\partial^2 \tilde{\tilde{\mathcal{L}}}}{\partial\phi^*\partial\phi\partial A_\kappa} = 2q^2\eta^{\mu\nu}\phi^*\phi = 2iq^2\eta^{\mu\nu}$$, where the factor 2 comes from the arbitrary (product rule on $A_\mu A_\nu$) contraction.

d) $\gamma(k_1)\,\gamma(k_2) \longrightarrow \phi^-(p_1)\,\phi^+(p_2)$, $k_1+k_2=p_1+p_2$

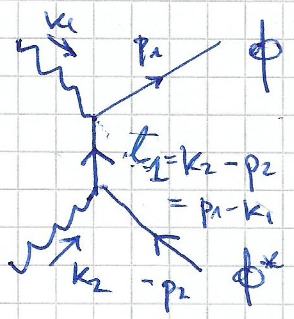
Are have arbitrary direction?



$$= \epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,(2iq^2 g^{\mu\nu}) = 2iq^2\,\epsilon_\mu(k_1)\,\epsilon^\mu(k_2)$$

$$\equiv i\mathcal{M}_1$$

Why 2 photons couple to 2 bosons here?



use antiparticles for bosons? But why $-k$ for them then?

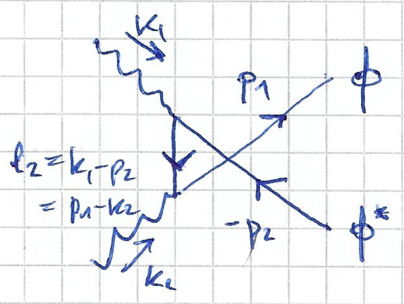
$$= \epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,(-iq)^2\,(l_1+p_1)^\mu (l_1-p_2)^\nu \frac{i}{l_1^2-m_\phi^2}$$

$$= -iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{l_1^2-m_\phi^2}\,(2p_1-k_1)^\mu (k_2-2p_2)^\nu$$

$$= 4iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{l_1^2-m_\phi^2}\,p_1^\mu p_2^\nu$$

in Lorentz gauge $\epsilon_1 \cdot k_1 = 0 = \epsilon_2 \cdot k_2$

$$\equiv i\mathcal{M}_2$$



Prop. w/ i or w/o. i?

$$= \epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,(-iq)^2\,(l_2-p_2)^\mu (l_2+p_1)^\nu \frac{i}{l_2^2-m_\phi^2}$$

$$= -iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{l_2^2-m_\phi^2}\,(k_1-2p_2)^\mu (2p_1-k_2)^\nu$$

required by Q.E.D. gauge inv.

$$= 4iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{l_2^2-m_\phi^2}\,p_2^\mu p_1^\nu$$

in Lorentz gauge

$$\equiv i\mathcal{M}_3$$

$$\rightsquigarrow i\mathcal{M}_{tot} = i\mathcal{M}_1 + i\mathcal{M}_2 + i\mathcal{M}_3 \quad , \quad t = l_1^2, \quad u = l_2^2$$

Considering CMS s.t. first diagram vanishes?

$$= 2iq^2\,\epsilon_\mu(k_1)\,\epsilon^\mu(k_2) + 4iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2) \left\{ \frac{p_1^\mu p_2^\nu}{t-m_\phi^2} + \frac{p_2^\mu p_1^\nu}{u-m_\phi^2} \right\}$$

Replacing $\epsilon_\mu(k_1) \longmapsto k_{1\mu}$, and noticing that

$$k_1 \cdot p_1 = \frac{1}{2}(m_\phi^2 - t) \quad \text{as} \quad (k_1-p_1)^2 = t$$

$$k_1 \cdot p_2 = \frac{1}{2}(m_\phi^2 - u) \quad \text{as} \quad (k_1-p_2)^2 = u$$

$$\rightsquigarrow i\mathcal{M}_{tot} = 2iq^2\,k_{1\mu}\,\epsilon^\mu(k_2) + 2iq^2\,\frac{m_\phi^2-t}{t-m_\phi^2}\,p_2^\nu\,\epsilon_\nu(k_2) + 2iq^2\,\frac{m_\phi^2-u}{u-m_\phi^2}\,p_1^\nu\,\epsilon_\nu(k_2)$$

$$\overset{p_1 = k_1+k_2-p_2}{\Rightarrow} = 2iq^2\,k_{1\mu}\,\epsilon^\mu(k_2) - 2iq^2 \left\{ p_2^\nu\,\epsilon_\nu(k_2) + (k_1+k_2-p_2)^\nu\,\epsilon_\nu(k_2) \right\}$$

$$= 2iq^2\,k_{1\mu}\,\epsilon^\mu(k_2) - 2iq^2\,k_1^\nu\,\epsilon_\nu(k_2) = 0$$

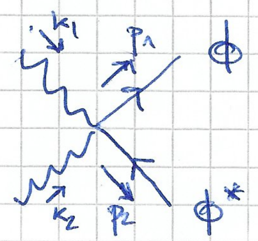
As $\mathcal{M} = \epsilon_\mu \mathcal{M}^\mu$; $k_\mu \mathcal{M}^\mu = 0$
$\epsilon^\mu \mapsto \epsilon^\mu + k^\mu$ $\epsilon_\mu \mathcal{M}^\mu$ L.I.

WRONG?

$\ell_{1/2}$ have arbitrary direction?

d) $\gamma(k_1)\,\gamma(k_2) \longrightarrow \phi^-(p_1)\,\phi^+(p_2)$, $k_1+k_2=p_1+p_2$

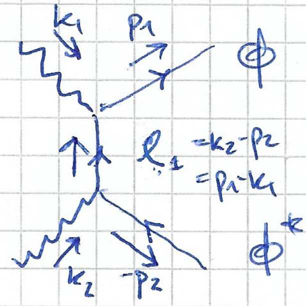
Why can 2 photons couple to 2 bosons here?



$$= \epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\left(2iq^2\eta^{\mu\nu}\right) = 2iq^2\,\epsilon_\mu(k_1)\,\epsilon^\mu(k_2)$$

$(*)$

use antiparticles for bosons?



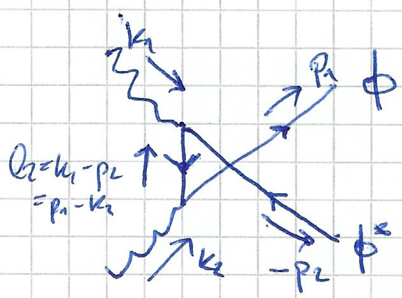
$$= \epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,(-iq)^2\,(\ell_1+p_1)^\mu\,(\ell_1-p_2)^\nu\,\frac{i}{\ell_1^2-m_\phi^2}$$

$$= -iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{\ell_1^2-m_\phi^2}\,(2p_1-k_1)^\mu\,\overbrace{(p_1-p_2-k_1)^\nu}^{k_2-2p_2}$$

$$= +4iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{\ell_1^2-m_\phi^2}\,p_1^\mu\,p_2^\nu$$

required by Q.E.D gauge invariance?

prop. i or $-i$?



$$= \epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,(-iq)^2\,(\ell_2-p_2)^\mu\,(\ell_2+p_1)^\nu\,\frac{i}{\ell_2^2-m_\phi^2}$$

$$= -iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{\ell_2^2-m_\phi^2}\,(k_1-2p_2)^\mu\,(2p_1-k_2)^\nu$$

$$= +4iq^2\,\epsilon_\mu(k_1)\,\epsilon_\nu(k_2)\,\frac{1}{\ell_2^2-m_\phi^2}\,p_2^\mu\,p_1^\nu$$

$(*)$ in CMS: $\quad k_1 = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}, \quad k_2 = \begin{pmatrix} k \\ 0 \\ 0 \\ -k \end{pmatrix}$

$$\epsilon^{(1)}(k_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)}(k_1) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\epsilon^{(1)}(k_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)}(k_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$\rightsquigarrow$ Replacing $\epsilon_\mu(k_1) \mapsto k_{1\mu}$

$\rightsquigarrow$ $2iq^2\,k_{1\mu}\,\epsilon^\mu(k_2) = 0$