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https://www.physics-and-stuff.com/

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22.11.2017 Theoretical Particle Physics Exercise 7

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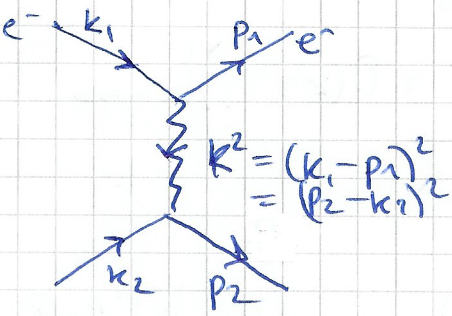
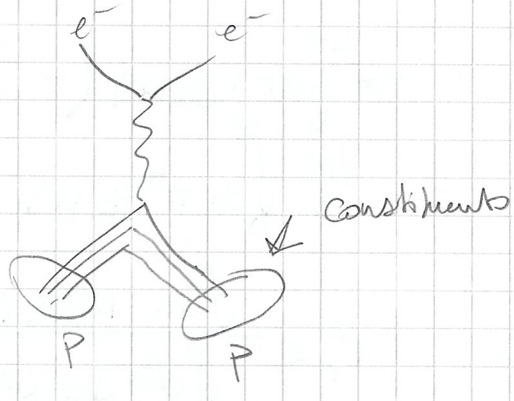
1)





What if direction of photon inverted? → $k = p_1 - k_1$ then? → yes, possible

a) $k^2 = |k^0|^2 - |\vec{k}|^2$

el. F.F ⇒ G_E or F_1? → G_E is the non-rel. limit of form-factor F_1

In the non-relativistic limit, we have $m \gg |\vec{p}|$ for all particles and their momenta.

We then consider:

$$k^2 = |k^0|^2 - |\vec{k}|^2 = |k_1^0 - p_1^0|^2 - |\vec{k}_1 - \vec{p}_1|^2$$

$$= \left(\sqrt{\vec{k}_1^2 + m_e^2} - \sqrt{\vec{p}_1^2 + m_e^2}\right)^2 - |\vec{k}_1 - \vec{p}_1|^2$$

$$= \left(m_e \sqrt{\left(\frac{\vec{k}_1}{m_e}\right)^2 + 1} - m_e \sqrt{\left(\frac{\vec{p}_1}{m_e}\right)^2 + 1}\right)^2 - |\vec{k}_1 - \vec{p}_1|^2$$

$$= \left(m_e \left(1 + \mathcal{O}\left(\frac{\vec{k}_1^2}{m_e^2}\right)\right) - m_e \left(1 + \mathcal{O}\left(\frac{\vec{p}_1^2}{m_e^2}\right)\right)\right)^2 - |\vec{k}_1 - \vec{p}_1|^2$$

What about $\mathcal{O}\left(\frac{k_1^2}{m_e^2}\right)$? Still no? → still small ~energy

$$= -|\vec{k}_1 - \vec{p}_1|^2 + \mathcal{O}\left(\frac{\vec{k}_1^2}{m_e}\right) + \mathcal{O}\left(\frac{\vec{p}_1^2}{m_e}\right) \approx -|\vec{k}|^2$$

b)

$$G_E(k^2) = \int d^3x \, \rho_Q(\vec{x}) e^{i\vec{k}\vec{x}}$$

Why only in this limit F.T. of charge distr.? → could also construct in rel. case

$$\overset{\text{sph. coord.}}{\underset{\rho_Q \text{ sph. sym.}}{=}} \int_0^\infty dr \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi \, \rho_Q(r) e^{i|\vec{k}| r \cos\theta} r^2$$

$$\overset{k = |\vec{k}|}{=} 2\pi \int_0^\infty dr \, r^2 \rho_Q(r) \left[\frac{-i}{kr} e^{ikr\cos\theta}\right]\Bigg|_{-1}^{1}$$

$$= 2\pi \int_0^\infty dr \, r \frac{\rho_Q(r)}{ik} \left(e^{ikr} - e^{-ikr}\right)$$

$$\overset{\frac{e^{ix} - e^{-ix}}{2i} = \sin x}{=} 2\pi \int_0^\infty dr \frac{r \rho_Q(r)}{k} \left(2 \sin(kr)\right)$$

$$= \frac{4\pi}{k} \int_0^\infty dr \, r \rho_Q(r) \sin(kr)$$

$$= \frac{4\pi}{k} \int_0^\infty dr \, r \rho_Q(r) \left(kr - \frac{k^3 r^3}{6} + \mathcal{O}(k^5 r^5)\right)$$

$$\rightsquigarrow \frac{dG_E(k^2)}{dk^2}\Big|_{k^2\to 0} = \frac{d}{dk^2}\left(\frac{4\pi}{k}\int_0^\infty dr\, r\rho_Q(r)\left(kr - \frac{k^3r^3}{6} + \mathcal{O}(k^5r^5)\right)\right)\Big|_{k^2\to 0}$$

$$= 4\pi\int_0^\infty dr\, r\rho_Q(r)\left(-\frac{r^3}{6} + \mathcal{O}(k^3r^5)\right)\Big|_{k^2\to 0}$$

$$= -\frac{2\pi}{3}\int_0^\infty dr\, r^4\rho_Q(r) \quad (*)$$

$k < 1$? $\mathcal{O}(k^3r^5)$? $\rightsquigarrow$ fine

What if only k dependence? Derivative w/ respect to k^2 $\rightsquigarrow \frac{1}{2|k|}\frac{\partial}{\partial|k|}$

c) Define: $r_p^2 \equiv \dfrac{\int d^3x\, |\vec{x}|^2\rho_Q(\vec{x})}{\int d^3x\, \rho_Q(\vec{x})} \equiv \dfrac{\int d^3x\, |\vec{x}|^2\rho_Q(\vec{x})}{Q}$

$$\rightsquigarrow r_p^2 \propto \int d^3x\, |\vec{x}|^2\rho_Q(\vec{x}) = 4\pi\int_0^\infty dr\, r^2 r^2\rho_Q(r)$$

$$= 4\pi\int_0^\infty dr\, r^4\rho_Q(r) = -6(*) \propto (*) \ (\hat{=}\ \mathrm{RHS})$$

Why is this the def. of the proton radius? Center of mass? $\rightsquigarrow$ theres also a rel. def.

d) $r_p^2 \propto \dfrac{dG_E(k^2)}{dk^2}\Big|_{k^2\to 0} \propto \dfrac{d}{dk^2}\dfrac{1}{\left(1-\frac{k^2}{\Lambda^2}\right)^2}\Big|_{k^2\to 0}$

$$= \frac{-2\left(1-\frac{k^2}{\Lambda^2}\right)\left(-\frac{1}{\Lambda^2}\right)}{\left(1-\frac{k^2}{\Lambda^2}\right)^4}\Big|_{k^2\to 0} = \frac{2}{\Lambda^2\left(1-\frac{k^2}{\Lambda^2}\right)^3}\Big|_{k^2\to 0} = \frac{2}{\Lambda^2}$$

μp lamb shift: $0{,}84087 \pm 0{,}00039$ fm

ep lamb shift: $0{,}8751 \pm 0{,}0061$ fm

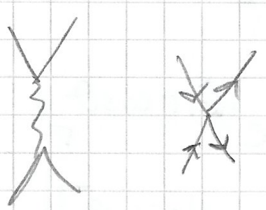
$0{,}03423$ fm

$\rightsquigarrow \Lambda \propto \frac{1}{r}$, Λ: energy scale up to which formfactor valid

$$\frac{1}{k^2}G_E(k^2) \approx \frac{1}{k^2}\left(1 + \frac{2k^2}{\Lambda^2}\right) \approx \frac{1}{k^2} + \frac{2}{\Lambda^2}$$

Contact term

point-like interaction doesn't depend on the ansatz



$\frac{1}{\Lambda}$ dim 5, $\frac{1}{\Lambda^2}$ dim 6

as $S = \int d^4x\, \mathcal{L}$

2)

a) $$\int_0^1 dx \int_0^1 dy \frac{1}{(Ax+By)^2}\delta(x+y-1)$$

$$= \int_0^1 dx \frac{1}{(xA+(1-x)B)^2} = \int_B^A dz \frac{1}{z^2}\frac{1}{A-B}$$

$$z = xA+(1-x)B$$

$$\rightsquigarrow \frac{dz}{dx} = (A-B)$$

$$= -\frac{1}{A-B}\left(\frac{1}{z}\right)\Big|_B^A = \frac{1}{B-A}\left(\frac{1}{A}-\frac{1}{B}\right) = \frac{1}{B-A}\left(\frac{B-A}{AB}\right)$$

$$= \frac{1}{AB}$$

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{2}{(Ax+By+Cz)^3}\delta(\overbrace{x+y+z-1}^{x+y+z=1})$$

$$= \int_0^1 dx \int_0^{1-x} dy \frac{2}{(xA+yB+(1-x-y)C)^3}$$

Why boundaries like this?

$\rightsquigarrow \delta(x+y+z-1)$
$z = 1-x-y$
$z=0,\ z=1$
$\rightsquigarrow z \geq 0, z \leq 1$
$\rightsquigarrow 1-x-y \geq 0$
$1-x-y \leq 1$
$\rightsquigarrow x+y \leq 1$
$x+y \geq 0$

$$W = xA+yB+(1-x-y)C$$

$$\rightsquigarrow \frac{dw}{dy} = B-C$$

$$= \int_0^1 dx \int_{xA+(1-x)C}^{xA+(1-x)B} dw \frac{2}{w^3}\frac{1}{B-C} = -\frac{1}{B-C}\int_0^1 dx \left(\frac{1}{w^2}\right)\Big|_{xA+(1-x)C}^{xA+(1-x)B}$$

$$= \frac{1}{C-B}\int_0^1 dx \left(\frac{1}{(xA+(1-x)B)^2} - \frac{1}{(xA+(1-x)C)^2}\right)$$

$$\overset{i)}{=} \frac{1}{C-B}\left\{\frac{1}{AB}-\frac{1}{AC}\right\} = \frac{1}{C-B}\left\{\frac{C-B}{ABC}\right\}$$

$$= \frac{1}{ABC}$$

b) For $n=1$, $\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \frac{1\cdot x^0}{(Ax+By)^2}\delta(x+y-1)$

was already proven in a) i).

So we claim that

$$\frac{1}{A^n B} = \int_0^1 dx \int_0^1 dy \frac{n x^{n-1}}{(Ax+By)^{n+1}}\delta(x+y-1)$$

Now: $n \mapsto n+1$ by deriving w/ respect to A:

$$\frac{1}{A^{n+1}B} = -\frac{1}{n}\frac{d}{dA}\frac{1}{A^n B} = -\frac{1}{n}\frac{d}{dA}\int_0^1 dx \int_0^1 dy \frac{n x^{n-1}}{(xA+yB)^{n+1}}\delta(x+y-1)$$

$$= -\frac{1}{n}\int_0^1 dx \int_0^1 dy \frac{n x^{n-1}}{(xA+yB)^{n+2}}\cdot(-(n+1))\cdot x\,\delta(x+y-1)$$

$$= \int_0^1 dx \int_0^1 dy \frac{(n+1)x}{(xA+yB)^{n+2}}\delta(x+y-1) \quad \boxtimes$$