

## Disclaimer

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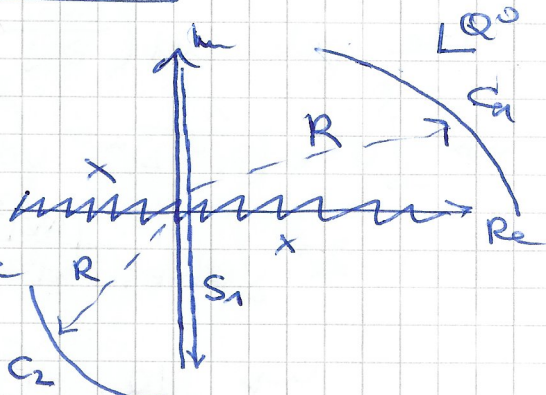
<https://www.physics-and-stuff.com/>

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1) a)  $I(\Delta, n) = \int_{-\infty}^{\infty} \frac{d^4 Q}{(2\pi)^4} \frac{1}{(Q^2 - \Delta)^n}$

Introducing an  $+i\epsilon$  prescription in the denominator, the poles are located somewhere as sketched.



We thus know from the residue theorem:

$$0 = I(\Delta, n) + \int_{S_1} \frac{d^4 Q}{(2\pi)^4} \frac{1}{(Q^2 - \Delta)^n} + \int_{C_1} \frac{d^4 Q}{(2\pi)^4} \frac{1}{(Q^2 - \Delta)^n} + \int_{C_2} \frac{d^4 Q}{(2\pi)^4} \frac{1}{(Q^2 - \Delta)^n}$$

where  $S_1$  is the path from  $+i\infty$  to  $-i\infty$  and  $C_1, C_2$  are the arcs sketched in the graph. The contribution from the arcs vanishes, as for  $n=1$ :

What about the  $Q^2$  in the denominator?  $\rightarrow$  take into the pole  $\Delta, (\Delta)$

$$I \sim \int_{C_1} dQ^0 \frac{1}{Q^0{}^2 - \Delta'} = \int_0^{2\pi} d\varphi \frac{1}{r^2 e^{2i\varphi} - \Delta'} r i e^{i\varphi} \quad (\Delta' = \Delta + \vec{Q}^2)$$

$Q^0 = r e^{i\varphi} \rightarrow Q^0(\varphi) = i r e^{i\varphi}$

$$\Rightarrow |I| \sim \left| \int_0^{2\pi} d\varphi \frac{i r e^{i\varphi}}{r^2 e^{2i\varphi} - \Delta'} \right| \leq \int_0^{2\pi} d\varphi \left| \frac{i r e^{i\varphi}}{r^2 e^{2i\varphi} - \Delta'} \right|$$

$$= \int_0^{2\pi} d\varphi \frac{r}{|r^2 e^{2i\varphi} - \Delta'|} \leq \int_0^{2\pi} d\varphi \frac{r}{|r^2 - \Delta'|}$$

$$|r^2 e^{2i\varphi} - \Delta'| \geq ||r^2 e^{2i\varphi}| - |\Delta'|| = |r^2 - \Delta'|$$

$$\leq \frac{\pi}{2} \max_{0 \leq \varphi \leq 2\pi} \frac{r}{|r^2 - \Delta'|} \xrightarrow{r \rightarrow \infty} 0$$

and thus:  $I(\Delta, n) = - \int_{S_1} \frac{d^4 Q}{(2\pi)^4} \frac{1}{(Q^2 - \Delta)^n}$

Parametrizing the imaginary way along the axis and reversing the direction of integration, we find:  $Q^0 = i Q_E^0, \vec{Q} = \vec{Q}_E$

$$\Rightarrow I(\Delta, n) = i \int_{-\infty}^{\infty} \frac{d^4 Q_E}{(2\pi)^4} \frac{1}{(-Q_E^0{}^2 - \vec{Q}_E^2 - \Delta)^n}$$

$$= \frac{(-1)^n i}{(2\pi)^4} \int_{-\infty}^{\infty} d^4 Q_E \frac{1}{(Q_E^2 + \Delta)^n}, \quad Q_E^2 = Q_E^0{}^2 + \vec{Q}_E^2$$

b) With  $Q_E = |Q_E| (\sin\omega \sin\phi \sin\theta, \cos\omega \sin\phi \sin\theta, \cos\phi \sin\theta, \cos\theta)$

we get  $\det\left(\frac{\partial Q_{Ei}}{\partial x_j}\right) = |Q_E|^3 \sin\phi \sin^2\theta$

$$\begin{aligned} \mapsto \int_{-\infty}^{\infty} d^4 Q_E \frac{1}{(Q_E^2 + \Delta)^n} &= \int_0^{\infty} d|Q_E| \int_0^{2\pi} d\omega \int_0^{\pi} d\phi \int_0^{\pi} d\theta |Q_E|^3 \sin\phi \sin^2\theta \frac{1}{(|Q_E|^2 + \Delta)^n} \\ &= 2\pi^2 \int_0^{\infty} d|Q_E| \frac{|Q_E|^3}{(|Q_E|^2 + \Delta)^n}, \text{ as } \int d\Omega_4 = \int_0^{2\pi} d\omega \int_0^{\pi} d\phi \int_0^{\pi} d\theta \sin\phi \sin^2\theta \end{aligned}$$

$$\begin{aligned} \mapsto \int d\Omega_4 &= (2\pi) \left( \int_0^{\pi} d\phi \sin\phi \right) \left( \int_0^{\pi} d\theta \sin^2\theta \right) \\ &= (2\pi) [-\cos\phi]_0^{\pi} \left[ \frac{\theta - \sin\theta \cos\theta}{2} \right]_0^{\pi} = 2\pi^2 \end{aligned}$$

Why is  $\omega$  that one going from 0 to  $2\pi$ ?  
 $\mapsto (\sin\omega, \cos\omega) \rightarrow$  from 0 to  $2\pi$

c)

$$\begin{aligned} I(\Delta, n) &= \frac{(-1)^n i}{(2\pi)^n} (2\pi^2) \int_0^{\infty} d|Q_E| \frac{|Q_E|^3}{(|Q_E|^2 + \Delta)^n} \\ &= \frac{(-1)^n i}{8\pi^2} \int_0^{\infty} dz \frac{|Q_E|^3}{2|Q_E|} \frac{1}{(z + \Delta)^n} \text{ using } z = |Q_E|^2 \\ &= \frac{(-1)^n i}{16\pi^2} \int_0^{\infty} dz \frac{z}{(z + \Delta)^n} \end{aligned}$$

$$\frac{dz}{d|Q_E|} = 2|Q_E|$$

What about  $n=2$ ?  
 $\mapsto$  pole see next sheet

$$= \frac{(-1)^n i}{16\pi^2} \left( \int_0^{\infty} dz \frac{1}{(z + \Delta)^{n-1}} - \Delta \int_0^{\infty} dz \frac{1}{(z + \Delta)^n} \right)$$

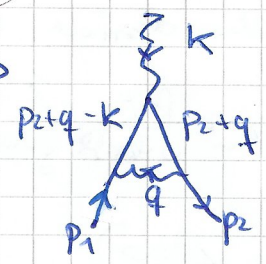
$$= \frac{(-1)^n i}{16\pi^2} \left( \frac{-1}{n-2} \frac{1}{(z + \Delta)^{n-2}} \Big|_0^{\infty} - \Delta \left( \frac{-1}{n-1} \right) \frac{1}{(z + \Delta)^{n-1}} \Big|_0^{\infty} \right)$$

$$\stackrel{n>2}{=} \frac{(-1)^n i}{16\pi^2} \left( \frac{1}{n-2} \Delta^{2-n} - \frac{\Delta}{n-1} \Delta^{1-n} \right)$$

$$= \frac{(-1)^n i}{16\pi^2} \left( \frac{(n-1) - (n-2)}{(n-1)(n-2)} \Delta^{2-n} \right) = \frac{(-1)^n i}{16\pi^2} \frac{\Delta^{2-n}}{(n-1)(n-2)}$$

✓  
 contribution to the wave fun.  
 renorm (mass)  
 renorm in fermion  
 case)

2) We considered the process  
 (vertex correction leading  
 order)



✓  
 Had  $\pm xyk^2$   
 in lecture?  
 $\rightarrow +$

and had 
$$I^\mu = 2e^2 \bar{u}(p_2) \int \frac{d^4 Q}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \frac{2mP^\mu (x+y - (x+y)^2) + y^2(-1)}{D^3} u(p_1)$$

where  $P^\mu = p_1^\mu + p_2^\mu$ ,  $D = Q^2 - m^2(x+y)^2 - xyk^2$   
 $k = p_2 - p_1$

there are  
 many  
 gordan decomp?

First, we derive the necessary gordan decomposition:  $(\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu])$

$$\begin{aligned} \bar{u}(p_2) i \sigma^{\mu\nu} k_\nu u(p_1) &= \bar{u}(p_2) \left(-\frac{i}{2}\right) (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) (p_{2\nu} - p_{1\nu}) u(p_1) \\ &= -\frac{i}{2} \bar{u}(p_2) \left\{ \gamma^\mu p_2^\nu - \gamma^\mu p_1^\nu - p_2^\nu \gamma^\mu + p_1^\nu \gamma^\mu \right\} u(p_1) \\ \stackrel{2\gamma^\mu \gamma^\nu = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu}{=} &= -\frac{i}{2} \bar{u}(p_2) \left\{ (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) p_{2\nu} - \gamma^\mu p_1^\nu - p_2^\nu \gamma^\mu + p_{1\nu} (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) \right\} u(p_1) \\ &= -\frac{i}{2} \bar{u}(p_2) \left\{ 2p_2^\mu - p_2^\nu \gamma^\mu - \gamma^\mu p_1^\nu - p_2^\nu \gamma^\mu + 2p_1^\mu - \gamma^\mu p_1^\nu \right\} u(p_1) \\ &= -\frac{i}{2} \bar{u}(p_2) \left\{ 2(p_2^\mu + p_1^\mu) - m\gamma^\mu - m\gamma^\mu - m\gamma^\mu - m\gamma^\mu \right\} u(p_1) \\ &= -i \bar{u}(p_2) \left\{ P^\mu - 2m\gamma^\mu \right\} u(p_1) \quad \text{using } (\not{p} - m)u(p) = 0 = \bar{u}(p)(\not{p} - m) \end{aligned}$$

Consider: 
$$2e^3 \bar{u}(p_2) \left\{ 2mP^\mu (x+y - (x+y)^2) \right\} u(p_1)$$
  

$$= 2e^3 \bar{u}(p_2) \left\{ 2m(x+y - (x+y)^2) \left[ -i \sigma^{\mu\nu} k_\nu + 2m\gamma^\mu \right] \right\} u(p_1)$$

As we want to achieve a form like

$$I^\mu = ie^2 \bar{u}(p_2) \left( \gamma^\mu F_1 + i \frac{\sigma^{\mu\nu} k_\nu}{2m} F_2 \right) u(p_1),$$

We will absorb the term above  $\sim \gamma^\mu$  in  $F_1$  in addition to  $(-)$ .

Defining 
$$F_2 = 2ie^2 (4m^2) \int \frac{d^4 Q}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y - (x+y)^2)}{D^3}$$
 we get

$$I_{F_2}^\mu = -2ie^3 \bar{u}(p_2) (2m) \sigma^{\mu\nu} k_\nu \int \frac{d^4 Q}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y - (x+y)^2)}{D^3} u(p_1)$$

Which is exactly the desired term.

b) With  $F_2 = 8ie^2m^2 \int_0^1 \frac{d^4Q}{(2\pi)^4} \int_0^{1-x} dx \int_0^{1-x} dy \frac{(x+y-(x+y)^2)}{(Q^2 - m^2(x+y)^2 - xyk^2)^3}$

we get (for  $k^2=0$ )

$F_2(0) = 8ie^2m^2 \int_0^1 dx \int_0^{1-x} dy (x+y-(x+y)^2) \int \frac{d^4Q}{(2\pi)^4} \frac{1}{(Q^2 - \Delta)^3}, \Delta = m^2(x+y)^2$

c)  $= 8ie^2m^2 \int_0^1 dx \int_0^{1-x} dy (x+y-(x+y)^2) \left( \frac{-i}{16\pi^2} \frac{\Delta^{-1}}{2} \right)$

$= \frac{e^2m^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y-(x+y)^2)}{m^2(x+y)^2}$

$= \frac{e^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{(x+y)}{(x+y)^2} - 1 \right\}$

$= \frac{e^2}{4\pi^2} \left\{ \int_0^1 dx \left[ \ln(1) - \ln(x) \right] - \int_0^1 dx (1-x) \right\}$

$= \frac{e^2}{4\pi^2} \left\{ - \left[ x(\ln x - 1) \right]_0^1 - \left[ 1 - \frac{1}{2} \right] \right\}$

$= \frac{e^2}{4\pi^2} \left\{ 1 - \frac{1}{2} \right\} = \frac{\alpha}{2} = 1,161714913 \cdot 10^{-3}$

$\Rightarrow g_e = 2 + 2F_2(0) = 2,00232343$

$\Delta_{\mu\nu} = 2,90 \cdot 10^{-3}$

$3,5 \cdot a_{\mu}^{exp} - a_{\mu}^{theory}$

✓  
Here already  
set  $k^2=0$ ?  
yes, calculating  
 $F_2(0)$ .  
No div.  
for  $F_1(k^2)$   
photo. not that  
easy.

✓  
Macroscopic  
ext mag  
field...  
Static photons  
no photon is  
on-shell; photo  
is propagating  
magnetic field  
(nucleus o.g.)  
mediates photon.  
 $\vec{B} = \nabla \times \vec{A}$  and  $\mathcal{L}_p$  in  
Lagrangian/Dirac

Why mass  
independent?  
Different for  
e and  $\mu$ ?  
no hadronic  
and  
Weak int. correction  
(bigger for  
muon than  
for electron)