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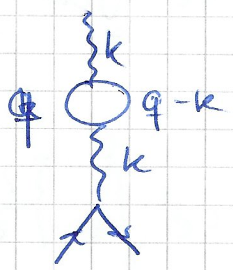
07-12-2017 Theoretical Particle Physics Homework 9

Marvin Zanke

1) $$\Pi^{\mu\nu}(k^2) = 4ie^2 \int \frac{d^4Q}{(2\pi)^4} \int_0^1 dx \, \frac{2Q^\mu Q^\nu - g^{\mu\nu}Q^2 + g^{\mu\nu}k^2x(1-x) + g^{\mu\nu}m^2}{[Q^2 + k^2x(1-x) - m^2]^2}$$

Why $Q^\mu Q^\nu \approx \frac{Q^2 g^{\mu\nu}}{d}$?



a) Using $Q^\mu Q^\nu \longrightarrow Q^2 g^{\mu\nu}/d$, we get (in d dimensions)

$$\Pi^{\mu\nu}(k^2) = 4ie^2 \int \frac{d^dQ}{(2\pi)^d} \int_0^1 dx \, \frac{2Q^2 g^{\mu\nu}/d - g^{\mu\nu}Q^2 + g^{\mu\nu}k^2x(1-x) + g^{\mu\nu}m^2}{[Q^2 + k^2x(1-x) - m^2]^2}$$

$$= 4ie^2 \int \frac{d^dQ}{(2\pi)^d} \int_0^1 dx \, \frac{\left(\frac{2}{d}-1\right)g^{\mu\nu}Q^2 + g^{\mu\nu}k^2x(1-x) + g^{\mu\nu}m^2}{[Q^2 + k^2x(1-x) - m^2]^2}$$

Wick: $Q^0 = iQ_E^0$, $\vec{Q} = \vec{Q}_E$

$$= -4e^2 \int \frac{d^dQ_E}{(2\pi)^d} \int_0^1 dx \, \frac{\left(1-\frac{2}{d}\right)g^{\mu\nu}Q^2 + g^{\mu\nu}k^2x(1-x) + g^{\mu\nu}m^2}{[Q_E^2 - k^2x(1-x) + m^2]^2}$$

$$= 4e^2 \int \frac{d^dQ_E}{(2\pi)^d} \int_0^1 dx \, \frac{\left(\frac{2}{d}-1\right)g^{\mu\nu}Q^2 - g^{\mu\nu}k^2x(1-x) - g^{\mu\nu}m^2}{[Q_E^2 + \Delta]^2}$$

w/ $\Delta = m^2 - k^2x(1-x)$

✓

For convergent integrals? $n > 2$ for HW 8 formula

b) In Homework 8, we had:

$$\int \frac{d^4Q_E}{(2\pi)^4} \frac{1}{(Q_E^2+\Delta)^n} = \frac{1}{16\pi^2} \frac{\Delta^{2-n}}{(n-1)(n-2)}$$

Now: $$\int \frac{d^dQ_E}{(2\pi)^d} \frac{1}{(Q_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \Delta^{d/2-n}$$

$$\overset{d=4,\ n>2}{=} \frac{1}{16\pi^2} \frac{\Gamma(n-2)}{\Gamma(n)} \Delta^{2-n} = \frac{1}{16\pi^2} \Delta^{2-n} \frac{(n-3)!}{(n-1)!}$$

$$= \frac{1}{16\pi^2} \frac{\Delta^{2-n}}{(n-2)(n-1)}$$

c) $$\Pi^{\mu\nu}(k^2) \overset{a)}{=} 4e^2 \int_0^1 dx \int \frac{d^dQ_E}{(2\pi)^d} \frac{\left(\frac{2}{d}-1\right)g^{\mu\nu}Q_E^2 - g^{\mu\nu}k^2x(1-x) - g^{\mu\nu}m^2}{[Q_E^2+\Delta]^2}$$

$$= 4e^2 \int_0^1 dx \left\{ \frac{2-d}{d} \int \frac{d^dQ_E}{(2\pi)^d} \frac{g^{\mu\nu}Q_E^2}{[Q_E^2+\Delta]^2} - g^{\mu\nu}[k^2x(1-x)+m^2] \times \int \frac{d^dQ_E}{(2\pi)^d} \frac{1}{[Q_E^2+\Delta]^2} \right\}$$

$$= 4e^2 \int_0^1 dx \left\{ \frac{2-d}{d} g^{\mu\nu} \left[\frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-d/2)}{\Gamma(2)} \Delta^{d/2-1} \right] - g^{\mu\nu}[k^2x(1-x)+m^2] \left[\frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \Delta^{d/2-2} \right] \right\}$$

$$= 4e^2 \int_0^1 dx \frac{g^{\mu\nu}}{(4\pi)^{d/2}} \left\{ \frac{2-d}{2} \Gamma(1-d/2)\Delta^{d/2-1} - (k^2x(1-x)+m^2)\Gamma(2-d/2)\Delta^{d/2-2} \right\}$$

$$= \frac{4e^2 g^{\mu\nu}}{(4\pi)^{d/2}} \int_0^1 dx \left\{ (1-d/2)\Gamma(1-d/2)\Delta^{d/2-1} - (k^2x(1-x)+m^2)\Gamma(2-d/2)\Delta^{d/2-2} \right\}$$

$$\equiv k^2 g^{\mu\nu} \Pi(k^2)$$

d) using $(1-d/2)\Gamma(1-d/2) = \Gamma(2-d/2)$, we get

$$\Pi^{\mu\nu}(k^2) = \frac{4e^2 g^{\mu\nu}}{(4\pi)^{d/2}} \int_0^1 dx \left\{ \Gamma(2-d/2)\Delta^{d/2-1} - (k^2x(1-x)+m^2)\Gamma(2-d/2)\Delta^{d/2-2} \right\}$$

$$= \frac{4e^2 g^{\mu\nu}\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 dx \left\{ \Delta^{d/2-1} - (k^2x(1-x)+m^2)\Delta^{d/2-2} \right\}$$

$$\overset{\Delta = m^2 - k^2x(1-x)}{=} \frac{4e^2 g^{\mu\nu}\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 dx \left\{ (m^2 - k^2x(1-x))\Delta^{d/2-2} - (k^2x(1-x)+m^2)\Delta^{d/2-2} \right\}$$

$$= \frac{4e^2 g^{\mu\nu}\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 dx \left\{ -2k^2x(1-x)\Delta^{d/2-2} \right\}$$

$$= \frac{-8e^2 g^{\mu\nu}}{(4\pi)^{d/2}} k^2\Gamma(2-d/2) \int_0^1 dx\, x(1-x)\Delta^{d/2-2}$$

e) Having $\Pi^{\mu\nu}(k^2) = \frac{-8e^2 g^{\mu\nu}}{(4\pi)^{d/2}} k^2 \Gamma(2-d/2) \int_0^1 dx\, x(1-x)\, \Delta^{d/2-2}$

Introduce mass Scale $e \to \tilde{e}\mu^{2-d/2}$ $\quad \equiv k^2 g^{\mu\nu} \Pi(k^2)$

$$\rightsquigarrow \Pi(k^2) = \frac{-8e^2}{(4\pi)^{d/2}} \Gamma(2-d/2) \int_0^1 dx\, x(1-x)\, \Delta^{d/2-2}$$

Why still divergent?

$$\overset{d=4-\epsilon}{(\to 4)} = -\frac{8e^2}{(4\pi)^{2-\epsilon/2}} \left(\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right) \int_0^1 dx\, x(1-x)\, \Delta^{-\epsilon/2}$$

$$= -\frac{8e^2}{16\pi^2} \left(\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right) \int_0^1 dx\, x(1-x) \left(\frac{4\pi}{\Delta}\right)^{\epsilon/2}$$

$$\left|\; x^{\epsilon/2} = \left(e^{\log x}\right)^{\epsilon/2} = e^{\frac{\epsilon}{2}\log x} = 1 + \frac{\epsilon}{2}\log x + \mathcal{O}(\epsilon^2)\right.$$

✓

Why does it correspond to the el. charge like this? resumming all the diagrams and $\hat{\Pi}(k^2) = \Pi(k^2) - \Pi(0)$ where $\Pi(0) = \ldots$ $\rightsquigarrow$ Difference to the non-corrected

$|k|^2 \gg m^2$ refers to 3-mom.? $\rightsquigarrow$ 4 momentum?

$$= \frac{-8e^2}{16\pi^2} \left(\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right) \int_0^1 dx\, x(1-x) \left(1 + \frac{\epsilon}{2}\log\frac{4\pi}{\Delta} + \mathcal{O}(\epsilon^2)\right)$$

$$= \frac{-8e^2}{16\pi^2} \int_0^1 dx\, x(1-x) \left\{\frac{2}{\epsilon} + \log\frac{4\pi}{\Delta} - \gamma_E + \mathcal{O}(\epsilon)\right\}$$

$$= \frac{-8e^2}{16\pi^2} \int_0^1 dx\, x(1-x) \left\{\frac{2}{\epsilon} - \log\Delta + \log 4\pi - \gamma_E + \mathcal{O}(\epsilon)\right\}$$

✓

f)

$\Delta = -\frac{k^2 x(1-x) + m^2}{m^2}$ but $x(1-x)$ can also be really small? $\rightsquigarrow$ maybe ≈ 0 at boundary // limit $\frac{k^2}{m^2} \to \infty$ faster than $x(1-x) \to 0$

$$\rightsquigarrow \hat{\Pi}(k^2) = \Pi(k^2) - \Pi(0) \quad (\text{only } k^2\text{-dep part is } \Delta)$$

$$= \frac{+8e^2}{16\pi^2} \left\{\int_0^1 dx\, x(1-x) \log\Delta - \int_0^1 dx\, x(1-x) \log\Delta\Big|_{k^2=0}\right\}$$

$$= \frac{8e^2}{16\pi^2} \left\{\int_0^1 dx\, x(1-x) \log\frac{m^2 - k^2 x(1-x)}{m^2}\right\}$$

$k^2 = 0$ before integrating?

$$\rightsquigarrow \hat{\Pi}(k^2) \xrightarrow{k^2 \to 0} 0$$

$$\hat{\Pi}(k^2) \xrightarrow{|k|^2 \gg m^2} \frac{8e^2}{16\pi^2} \int_0^1 dx\, x(1-x) \log\frac{-k^2 x(1-x)}{m^2}$$

$$= \frac{8e^2}{16\pi^2} \int_0^1 dx\, x(1-x) \left\{\log\frac{-k^2}{m^2} + \log(x(1-x))\right\}$$

$$\alpha = \frac{e^2}{4\pi} \qquad = \frac{8e^2}{16\pi^2} \left\{\frac{1}{6}\log\frac{-k^2}{m^2} - \frac{5}{18}\right\} = \frac{e^2}{12\pi^2} \left\{\log\frac{-k^2}{m^2} - \frac{5}{3}\right\}$$

$$= \frac{\alpha}{3\pi} \left\{\log\frac{-k^2}{m^2} - \frac{5}{3}\right\}$$

$$\alpha_{eff}(k^2) = \frac{\alpha^2}{1 - \frac{\alpha}{3\pi}\left[\log\left(-\frac{k^2}{m^2}\right) - 5/3\right]}$$

2)

a) For an arbitrary X_a $(, X_b, X_c)$, we have

$$[X_a,[X_b,X_c]]+[X_b,[X_c,X_a]]+[X_c,[X_a,X_b]]$$

Faster way? ⇝ Not done in tutorial

$$= X_a[X_b,X_c]-[X_b,X_c]X_a+X_b[X_c,X_a]-[X_c,X_a]X_b + X_c[X_a,X_b]-[X_a,X_b]X_c$$

$$= X_a(X_bX_c - X_cX_b) - (X_bX_c - X_cX_b)X_a + X_b(X_cX_a - X_aX_c) - (X_cX_a - X_aX_c)X_b + X_c(X_aX_b - X_bX_a) - (X_aX_b - X_bX_a)X_c$$

$= 0$, which is called the Jacobi Identity

It's "generators" – not the generators of the group?

b) Define $[T_a, T_b] \equiv i f^{abc} T_c$ for the generators of the Algebra, where $tr\, T_a T_b = \frac{1}{2}\delta_{ab}$.

Structure constants

We have:

why $tr\, T_a T_b \neq 0$ for $a=b$?

$$[T_a,T_b] = i f^{abc} T_c$$
$$= -[T_b,T_a] = -i f^{bac} T_c$$

⇝ $f^{abc} = -f^{bac}$ ⇝ antisym. in the first 2 indices

Now, consider

$[T_a,T_b] = i f^{abc} T_c$ and multiply w/ T_d from the left

⇝ $T_d[T_a,T_b] = i f^{abc} T_d T_c$, take the trace now

⇝ $tr(T_d[T_a,T_b]) = \frac{i}{2} f^{abd}$

$tr(A(BC-CB))$
$= tr(BCA - BAC)$
$= tr(B(CA-AC))$

$= tr(T_b[T_d,T_a]) = -tr(T_b[T_a,T_d])$

$= -tr(T_b\, i f^{adc} T_c) = -\frac{i}{2} f^{adb}$

⇝ $f^{abd} = -f^{adb}$ ⇝ antisymmetric in last 2 indices.

It follows: $f^{abc} = -f^{bac} = f^{bca} = -f^{cba}$ ⇝ antisym. in first and last index.

c) Using the matrices $(T_a)^{bc} := -if^{abc}$, we want to show that they fulfill the algebra.

" furnish" a repr. of the algebra?

In general, one finds for the structure constants, using the Jacobi identity and the generator relation:

$$0 = [T_a,[T_b,T_c]] + [T_b,[T_c,T_a]] + [T_c,[T_a,T_b]]$$

$$= [T_a, if^{bck}T_k] + [T_b, if^{cak}T_k] + [T_c, if^{abk}T_k]$$

$$= (if^{bck})(if^{akl}T_l) + (if^{cak})(if^{bkl}T_l)$$

$$+ (if^{abk})(if^{ckl}T_l)$$

$$= (-f^{bck}f^{akl} - f^{cak}f^{bkl} - f^{abk}f^{ckl})T_l$$

$$= (f^{bck}f^{alk} + f^{cak}f^{blk} + f^{abk}f^{clk})T_l$$

$$\rightsquigarrow f^{bck}f^{alk} + f^{cak}f^{blk} + f^{abk}f^{clk} = 0$$

What are the T's in (8) then? the $(T_a)^{bc}$'s are the generators? ⤳ Better call M or sth.

the generators still exist ∀ and eq. still holds.

What is this adjoint algebra?

Why does already have to hold for the f's w/o the T_l?

Now consider the given matrix representation:

$$[T_a,T_b]_{xy} = (T_a)_{xk}(T_b)_{ky} - (T_b)_{xk}(T_a)_{ky}$$

$$= (-if^{axk})(-if^{bky}) - (-if^{bxk})(-if^{aky})$$

$$= -f^{axk}f^{bky} + f^{bxk}f^{aky}$$

$$= f^{axk}f^{byk} + f^{bxk}f^{aky}$$

$$= -(f^{xbk}f^{ayk} + f^{bak}f^{xyk}) + f^{xbk}f^{ayk}$$

$$= -f^{bak}f^{xyk} = f^{abk}f^{xyk} = if^{abk}(T_k)_{xy}$$

$$\forall xy \rightsquigarrow [T_a,T_b] = if^{abk}T_k$$