Hinweis

Die vorliegende Lösung wurde im Rahmen der jeweiligen Lehrveranstaltung an der Universität Bonn erstellt. Sofern im oberen Teil der ersten Seite oder auf der unten angegebenen Webseite nicht anders vermerkt, wurde diese Lösung von mir, Marvin Zanke, alleine angefertigt und eingereicht. Bei allem in einer anderen Farbe als dem üblichen Blau handelt es sich in der Regel um Korrekturen von mir oder des Tutors. Für mehr Informationen und meine gesamten Unterlagen, siehe:
https://www.physics-and-stuff.com/

Ich erhebe keinen Anspruch auf Richtigkeit und Vollständigkeit der vorliegenden Lösungen! Dies gilt ebenso für obengenannte Korrekturen.

Theoretische Physik I Blatt 11

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H1	H2	Σ
9.5	9.5	19

H1) a) $H = \sum_{i=1}^{n} \dot{q}_i p_i - L \quad , H(p,q,t)$

$$dH = \sum_{i=1}^{n} \dot{q}_i dp_i + p_i d\dot{q}_i - dL$$

$$= \sum_{i=1}^{n} \left(\dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) - \frac{\partial L}{\partial t} dt$$ ✓

Außerdem:

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right) + \frac{\partial H}{\partial t} dt$$ ✓

Vergleich ergibt:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad ✓ \quad -\frac{\partial L}{\partial q_i} = -\dot{p}_i = \frac{\partial H}{\partial q_i} \quad ✓$$

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ (kennen wir bereits, ist der kanonische Impuls.)

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad (*)$$ ✓

ELG / n Gleichungen, Grad 2, d.h. es kommen Ableitungen zweiten Grades vor. Symmetrisch (mit "-") bzgl p_i und q_i bei Hamilton und 2n Gleichungen: Grad 1. ✓ 3/3

b) $$\frac{dH}{dt} = \sum_{i=1}^{n} \left(\ddot{q}_i p_i + \dot{p}_i \dot{q}_i \right) - \frac{dL}{dt}$$

$$= \sum_{i=1}^{n} \left(\ddot{q}_i p_i + \dot{p}_i \dot{q}_i - \underbrace{\frac{\partial L}{\partial q_i}}_{\dot{p}_i} \dot{q}_i - \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \ddot{q}_i \right) - \frac{\partial L}{\partial t}$$

$$= \sum_{i=1}^{n} 0 - \frac{\partial L}{\partial t}$$

$$\overset{(*)}{=} \frac{\partial H}{\partial t}$$ ✓

H ist erhalten genau dann wenn H nicht expizit von der Zeit abhängt. ✓ 1/1

c) Für zyklische Koordinaten gilt $p_i = \frac{\partial L}{\partial \dot{q}_i} = \text{const}$

$\Rightarrow \dot{p}_i = 0 \Rightarrow \frac{\partial H}{\partial q_i} = 0$ ✓ 1/1

d) In einem konservativen System gilt $H = T + V = E$
also Gesamtenergie

$\frac{1}{1}$

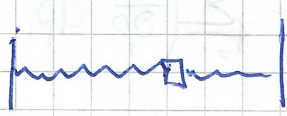
e) $0 = \delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i^1} \cdot \delta q^1 + \frac{\partial H}{\partial q_i^2} \delta q^2 + \frac{\partial H}{\partial q_i^3} \delta q^3$

Da Variation für alle δq^j separat $\Rightarrow \sum_i \frac{\partial H}{\partial q_i^j} = \sum_i -\dot{p}_i^j = 0 \ \forall j$

$\Rightarrow \sum_i p_i^j = \text{const} \quad \forall j = 1,2,3$

$\frac{1}{1}$

f) $H = \sum_{i=1}^{n} \dot{q}_i p_i - L$



harm. Ost.: $L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{k}{2} x^2$, $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$\Rightarrow H = m\dot{x}^2 - \frac{1}{2} m \dot{x}^2 + \frac{k}{2} x^2$

$= \frac{1}{2} m \dot{x}^2 + \frac{k}{2} x^2 = \frac{p^2}{2m} + \frac{k}{2} x^2$

$\frac{\partial H}{\partial q_i} = -\dot{p}_i$, $\frac{\partial H}{\partial p_i} = \dot{q}_i \Leftrightarrow \frac{\partial H}{\partial x} = -\dot{p}$, $\frac{\partial H}{\partial p} = \dot{x}$

$-\dot{p} = \frac{\partial H}{\partial x} = kx$, $\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \to \ddot{x} = \frac{\dot{p}}{m} = \frac{-kx}{m}$

Kepler-Problem: $L = T - V$, $V = +\frac{\alpha}{r}$, $T = \frac{1}{2} m \dot{\vec{r}}^2$

Zylinderkoord.: $\vec{r}(t) = \begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix}$, $\dot{\vec{r}}(t) = \begin{pmatrix} \dot{\rho}\cos\varphi - \rho\sin\varphi\dot{\varphi} \\ \dot{\rho}\sin\varphi + \rho\cos\varphi\dot{\varphi} \\ \dot{z} \end{pmatrix}$

$\Rightarrow L = \frac{1}{2} m \big(\dot{\rho}^2 \cos^2\varphi + \rho^2 \sin^2\varphi \dot{\varphi}^2 - 2\dot{\rho}\rho \cos\varphi\sin\varphi\dot{\varphi}$

$+ \dot{\rho}^2 \sin^2\varphi + \rho^2 \cos^2\varphi \dot{\varphi}^2 + 2\dot{\rho}\rho\cos\varphi\sin\varphi\dot{\varphi} + \dot{z}^2 \big) + \frac{\alpha}{r}$

$= \frac{1}{2} m (\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) + \frac{\alpha}{r}$ $\quad p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$, $\frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} = p_\varphi$

$H = \frac{p_\rho^2}{m} + \frac{p_\varphi^2}{m\rho^2} + \frac{p_z^2}{m} - \frac{1}{2}m\left(\frac{p_\rho^2}{m^2} + \rho^2 \frac{p_\varphi^2}{m^2\rho^4} + \frac{p_z^2}{m^2}\right) - \frac{\alpha}{r}$ $\quad \frac{\partial L}{\partial \dot{z}} = m\dot{z} = p_z$

$= \frac{p_\rho^2}{2m} + \frac{p_\varphi^2}{2m\rho^2} + \frac{p_z^2}{2m} - \frac{\alpha}{\rho}$

$-\dot{p}_\rho = \frac{\partial H}{\partial \rho} = \frac{\alpha}{\rho^2} - \frac{p_\varphi^2}{m\rho^3}$, $\dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} \Rightarrow \ddot{\rho} = -\frac{\alpha}{m\rho^2} + \frac{p_\varphi^2}{m^2\rho^3} = -\frac{\alpha}{m\rho^2} + \frac{m^2\rho^4\dot{\varphi}^2}{m^2\rho^3}$

$-\dot{p}_\varphi = \frac{\partial H}{\partial \varphi} = 0$, $\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\rho^2} \Rightarrow \ddot{\varphi} = 0$ $\qquad \ddot{\rho} = -\frac{\alpha}{m\rho^2} + \rho\dot{\varphi}^2$

$-\dot{p}_z = \frac{\partial H}{\partial z} = 0$, $\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \Rightarrow \ddot{z} = 0$

$\frac{2.5}{3}$

H2)
a) $\{f, g\} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$

$$\{q_i, q_j\} = \sum_k \frac{\partial q_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial q_j}{\partial q_k}$$

$$= \sum_k \delta_{ik} \cdot 0 - 0 \cdot \delta_{jk} = 0 \checkmark$$

$$\{p_i, p_j\} = \sum_k \frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k}$$

$$= \sum_k 0 \cdot \delta_{jk} - \delta_{ik} \cdot 0 = 0 \checkmark$$

$$\{q_i, p_j\} = \sum_k \frac{\partial q_i}{\partial q_k} \cdot \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k}$$

$$= \sum_k \delta_{ik} \cdot \delta_{jk} - 0 \cdot 0 = \delta_{ij} \checkmark$$

$\frac{2}{2}$

b) $-\{f, q_i\} = -\sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_i}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial q_i}{\partial q_k}$

$$= -\sum_k \frac{\partial f}{\partial q_k} \cdot 0 - \frac{\partial f}{\partial p_k} \cdot \delta_{ik}$$

$$= + \frac{\partial f}{\partial p_i} \checkmark$$

$$\{f, p_i\} = \sum_k \frac{\partial f}{\partial q_k} \frac{\partial p_i}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial p_i}{\partial q_k}$$

$$= \sum_k \frac{\partial f}{\partial q_k} \delta_{ik} - \frac{\partial f}{\partial p_k} \cdot 0$$

$$= \frac{\partial f}{\partial q_i} \checkmark$$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i + \frac{\partial f}{\partial t}$$

Nun gilt wie wir wissen: $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Damit folgt: $\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial t}$

$$= \{f, H\} + \frac{\partial f}{\partial t} \checkmark$$

$\frac{3}{3}$

Für die Bew.gl. gilt dann: $\dot{q}_i = -\{H, q_i\} = \{q_i, H\} \checkmark$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\{H, p_i\} = \{p_i, H\} \checkmark$$

c) $\{fg, h\} = \sum_i \frac{\partial(fg)}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial(fg)}{\partial p_i}\frac{\partial h}{\partial q_i}$

$= \sum_i \left(g\frac{\partial f}{\partial q_i} + f\frac{\partial g}{\partial q_i}\right)\frac{\partial h}{\partial p_i} - \left(g\frac{\partial f}{\partial p_i} + f\frac{\partial g}{\partial p_i}\right)\frac{\partial h}{\partial q_i}$

$= g\sum_i \frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i} + f\sum_i \frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}$

$= g\{f,h\} + f\{g,h\}$ ✓ Warum so kompliziert?

$\frac{d}{dt}\{f,g\} - \{\{f,g\},H\} = \frac{d}{dt}\{f,g\} - \frac{d}{dt}\{f,g\} + \frac{\partial}{\partial t}\{f,g\}$

$= \frac{\partial}{\partial t}\{f,g\} = \sum_i \frac{\partial^2 f}{\partial t\partial q_i}\frac{\partial g}{\partial p_i} + \frac{\partial^2 g}{\partial t\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial^2 f}{\partial t\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial^2 g}{\partial t\partial q_i}\frac{\partial f}{\partial p_i}$

$= \sum_i \frac{\partial^2 f}{\partial t\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial^2 f}{\partial t\partial p_i}\frac{\partial g}{\partial q_i} + \sum_i \frac{\partial^2 g}{\partial t\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial^2 g}{\partial t\partial q_i}\frac{\partial f}{\partial p_i}$

$= \left\{\frac{\partial}{\partial t}f, g\right\} + \left\{f, \frac{\partial}{\partial t}g\right\}$ ✓ $\square$

$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\}$

$= \sum_i \frac{\partial\{f,g\}}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial\{f,g\}}{\partial p_i}\frac{\partial h}{\partial q_i} + \sum_i \frac{\partial\{h,f\}}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial\{h,f\}}{\partial p_i}\frac{\partial g}{\partial q_i}$

$+ \sum \frac{\partial\{g,h\}}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial\{g,h\}}{\partial p_i}\frac{\partial f}{\partial q_i}$

$\cancel{= \sum_i \frac{\partial^2 f}{\partial q_i^2}\frac{\partial g}{\partial p_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial q_i\partial p_i}\frac{\partial h}{\partial p_i} - \sum_i \frac{\partial^2 f}{\partial p_i\partial q_i}\frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}} -$

$= \sum_i \left(\frac{\partial^2 f}{\partial q_i^2}\frac{\partial g}{\partial p_i} + \frac{\partial^2 g}{\partial q_i\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial^2 f}{\partial q_i\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial^2 g}{\partial q_i^2}\frac{\partial f}{\partial p_i}\right)\frac{\partial h}{\partial p_i}$

$- \sum_i \left(\frac{\partial^2 f}{\partial p_i\partial q_i}\frac{\partial g}{\partial p_i} + \frac{\partial^2 g}{\partial p_i^2}\frac{\partial f}{\partial q_i} - \frac{\partial^2 f}{\partial p_i^2}\frac{\partial g}{\partial q_i} - \frac{\partial^2 g}{\partial p_i\partial q_i}\frac{\partial f}{\partial p_i}\right)\frac{\partial h}{\partial q_i}$

$+ \sum \left(\frac{\partial^2 h}{\partial q_i^2}\frac{\partial f}{\partial p_i} + \frac{\partial^2 f}{\partial q_i\partial p_i}\frac{\partial h}{\partial q_i} - \frac{\partial^2 h}{\partial q_i\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial^2 f}{\partial q_i^2}\frac{\partial h}{\partial p_i}\right)\frac{\partial g}{\partial p_i}$

$- \sum \left(\frac{\partial^2 h}{\partial p_i\partial q_i}\frac{\partial f}{\partial p_i} + \frac{\partial^2 f}{\partial p_i^2}\frac{\partial h}{\partial q_i} - \frac{\partial^2 h}{\partial p_i^2}\frac{\partial f}{\partial q_i} - \frac{\partial^2 f}{\partial p_i\partial q_i}\frac{\partial h}{\partial p_i}\right)\frac{\partial g}{\partial q_i}$

$+ \sum \left(\frac{\partial^2 g}{\partial q_i^2}\frac{\partial h}{\partial p_i} + \frac{\partial^2 h}{\partial q_i\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial^2 g}{\partial q_i\partial p_i}\frac{\partial h}{\partial q_i} - \frac{\partial^2 h}{\partial q_i^2}\frac{\partial g}{\partial p_i}\right)\frac{\partial f}{\partial p_i}$

$- \sum \left(\frac{\partial^2 g}{\partial q_i\partial p_i}\frac{\partial h}{\partial p_i} + \frac{\partial^2 h}{\partial p_i^2}\frac{\partial g}{\partial q_i} - \frac{\partial^2 g}{\partial p_i^2}\frac{\partial h}{\partial q_i} - \frac{\partial^2 h}{\partial p_i\partial q_i}\frac{\partial g}{\partial p_i}\right)\frac{\partial f}{\partial q_i}$

✓ 2/2

d) $l_i = \epsilon_{ijk} q_j p_k \Rightarrow \vec{L} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} q_2 p_3 - q_3 p_2 \\ q_3 p_1 - q_1 p_3 \\ q_1 p_2 - q_2 p_1 \end{pmatrix}$

$$\{l_i, q_j\} = \sum_k \frac{\partial l_i}{\partial q_k} \frac{\partial q_j}{\partial p_k} - \frac{\partial l_i}{\partial p_k} \frac{\partial q_j}{\partial q_k}$$

$$= \sum_k \frac{\partial l_i}{\partial q_k} \cdot 0 - \delta_{jk} \frac{\partial l_i}{\partial p_k} = -\frac{\partial l_i}{\partial p_j} = \epsilon_{ijm} q_m$$

$$\{l_i, p_j\} = \sum_k \frac{\partial l_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial l_i}{\partial p_k} \cdot \frac{\partial p_j}{\partial q_k}$$

$$= \sum_k \frac{\partial l_i}{\partial q_k} \cdot \delta_{jk} - \frac{\partial l_i}{\partial p_k} \cdot 0 = \frac{\partial l_i}{\partial q_j} = \epsilon_{jim} p_m$$

$$\{l_i, l_j\} = \epsilon_{ikl} \{q_k p_l, l_j\} = \epsilon_{ikl} \left[q_k \{p_l, l_j\} + p_l \{q_k, l_j\} \right] \quad (*)$$

Nun gilt: $\{p_l, l_j\} = \epsilon_{jmn} \{p_l, q_m p_n\} = \epsilon_{jmn} \{p_l, q_m\} p_n$

$$= -\epsilon_{jmn} p_n \delta_{lm}$$

$$= -\epsilon_{jln} p_n$$

und $\{q_k, l_j\} = \epsilon_{jmn} \{q_k, q_m p_n\} = \epsilon_{jmn} \{q_k, p_n\} q_m$

$$= \epsilon_{jmn} q_m \delta_{kn}$$

$$= \epsilon_{jmk} q_m$$

Damit folgt

$$(*) = \epsilon_{ikl} \left[q_k \epsilon_{jln} p_n + p_l \epsilon_{jmk} q_m \right]$$

Kontraktion ϵ-Tensor: $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

$$= \epsilon_{ikl} \left[q_k \epsilon_{ljn} p_n \right] - \epsilon_{ilk} \left[p_l \epsilon_{kmj} q_m \right]$$

$$= q_k p_n \left[\delta_{ij} \delta_{kn} - \delta_{in} \delta_{kj} \right] + p_l q_m \left[\delta_{im} \delta_{lj} - \delta_{ij} \delta_{lm} \right]$$

$$= \delta_{ij} q_k p_k - \delta_{ij} p_l q_l + p_j q_i - q_j p_i$$

$$= q_i p_j - q_j p_i = \epsilon_{ijk} l_k$$

$\{l_i, |\vec{q}|^2\} = \{l_i, q_1^2+q_2^2+q_3^2\} = \{l_i, q_1^2\} + \{l_i, q_2^2\} + \{l_i, q_3^2\}$

$= 2|\vec{q}|\{l_i, |\vec{q}|\} = 2q_1\{l_i, q_1\} + 2q_2\{l_i, q_2\} + 2q_3\{l_i, q_3\}$

$= -2q_1 \frac{\partial l_i}{\partial p_1} - 2q_2 \frac{\partial l_i}{\partial p_2} - 2q_3 \frac{\partial l_i}{\partial p_3}$

$\Rightarrow \{l_i, |\vec{q}|\} = \frac{\{l_i, |\vec{q}|^2\}}{2|\vec{q}|} = -\frac{q_1}{|\vec{q}|}\frac{\partial l_i}{\partial p_1} - \frac{q_2}{|\vec{q}|}\frac{\partial l_i}{\partial p_2} - \frac{q_3}{|\vec{q}|}\frac{\partial l_i}{\partial p_3}$ f

$\{l_i, l^2\} = \{l_i, l_1^2 + l_2^2 + l_3^2\} = \{l_i, l_1^2\} + \{l_i, l_2^2\} + \{l_i, l_3^2\}$

$= 2l_1\{l_i, l_1\} + 2l_2\{l_i, l_2\} + 2l_3\{l_i, l_3\}$

$= 2l_1 \epsilon_{i1k} l_k + 2l_2 \epsilon_{i2k} l_k + 2l_3 \epsilon_{i3k} l_k$

$= 2\epsilon_{ijk} l_j l_k = 2(\vec{l} \times \vec{l}) = 0$ ✓

$\frac{2.5}{3}$