Disclaimer

The notes at hand were written during my research period as a PhD student at the University of Bonn. They contain auxiliary calculations to and comments on publications by other authors, which are subject to definite conditions of use; see also the respective article(s) on https://arxiv.org/ linked on the following website. For more information and all my material, check:
https://www.physics-and-stuff.com/

I raise no claim to correctness and completeness of the given material!

Note that the license of the respective article(s) on https://arxiv.org/ applies. Do not cite these notes but the original authors and the published article as given on https://www.physics-and-stuff.com/.

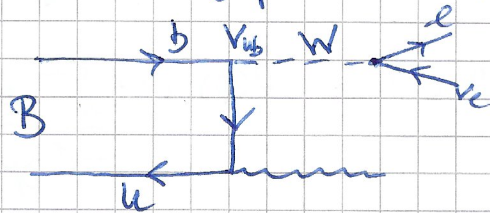
$B \to \ell \nu \gamma$ Decays (Khodjamirian et al.) 1

26.07.2021 We confirm some calculations from <0811.2487> (Khodjamirian et al.).

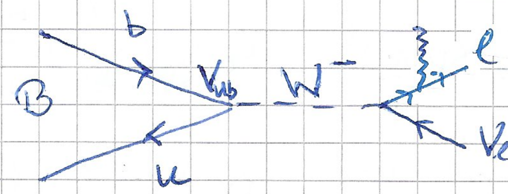
The amplitude for $B \to \ell \nu \gamma$ can be written as

$$A(B^- \to \ell \bar{\nu}_\ell \gamma) = \frac{G_F}{\sqrt{2}} V_{ub} \langle \ell \bar{\nu}(p) \gamma(q) | (\bar{\ell} \Gamma_\delta \nu)(\bar{u} \Gamma^\delta b) | B^-(p+q) \rangle$$

where $\Gamma_\alpha = \gamma_\alpha (1 - \gamma_5)$.



To first order in the e.m. interactions, the matrix element can be rewritten as a sum of two physically distinct contributions (see the diagrams on the top/right).



Why the factor of i? One for each photon current, right? And why the additional sign?

$$\langle \ell \bar{\nu}(p) \gamma(q) | (\bar{\ell} \Gamma_\delta \nu)(\bar{u} \Gamma^\delta b) | B^-(p+q) \rangle$$

$$= i e \epsilon^\rho \Big[(\bar{u}_\ell \Gamma_\delta v_\nu) \int d^4x \, e^{iq \cdot x} \langle 0 | T\{ j_\rho^{em}(x) \bar{u} \Gamma^\delta b(0) \} | B^-(p+q) \rangle$$

update (27.07.2021) note that

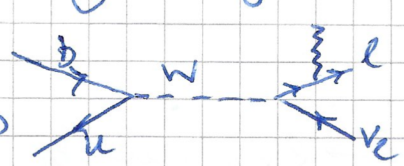
$$- \int d^4x \, e^{iq\cdot x} \langle \ell \bar{\nu}(p) | T\{ j_\rho^{em}(x) \bar{\ell} \Gamma_\delta \nu(0) \} | 0 \rangle \, i f_B (p+q)^\delta \quad (1)$$

outgoing photons are described by ϵ^{*}; wrong throughout the whole calculation

where $j_\rho^{em} = - \bar{\ell} \gamma_\rho \ell + \sum_{q=u,d,s,c,b} e_q \bar{q} \gamma_\rho q$ is the e.m. current and e_q the quark e.m. charges in units of e. The first term on the RHS corresponds to the photon emission from the initial B meson state, where the leptonic part is trivially factorized out. The second term on the RHS corresponds to photon emission from the final charged lepton and the hadronic matrix element is factorized using the standard definition of the B-meson decay constant,

What about the vector part $\langle 0 | \bar{u} \gamma^\delta b | B(p+q) \rangle$ here? B pseudoscalar?! And $\langle 0 | \bar{u} \gamma_5 b | B(p+q) \rangle$?

$$\langle 0 | \bar{u} \gamma^\delta \gamma_5 b | B(p+q) \rangle = i f_B (p+q)^\delta .$$

The remaining lepton-photon matrix element in this term can be calculated using the Feynman rules of QED (note the propagator in ), which, in the chiral limit $m_\ell = 0$ yields

$$ie\,\epsilon^\mu \Big[-\int d^4x\, e^{iq\cdot x} \langle \ell\bar\nu(p)|T\{j_\mu^{em}(x)\,\bar\ell\,\Gamma_S\, \nu(0)\}|0\rangle\, i f_B\,(p+q)^S\Big]$$

$$= e\,\epsilon^\mu \int d^4x\, e^{iq\cdot x} \langle \ell\bar\nu(p)|T\{\underbrace{j_\mu^{em}}_{-\bar\ell\gamma_\mu\ell}(x)\,\bar\ell\,\Gamma_S\, \nu(0)\}|0\rangle\, f_B\,(p+q)^S$$

$$= -e\,\epsilon^\mu f_B\,(p+q)^S \Big[\bar u_\ell\, \gamma_\mu \frac{i(\not p_\ell+\not q)}{(p_\ell+q)^2}\, \gamma_S(1-\gamma_5)\, v_\nu\Big]$$

$$\Big|\quad p = p_\ell + p_\nu \quad\text{and}\quad \not p_\nu v_\nu = 0\;;\; \frac{\not p_\ell+\not q}{(p_\ell+q)^2} \mathrel{\hat=} \frac{1}{\not p_\ell+\not q}$$

$$= -ie\,\epsilon^\mu f_B\, \bar u_\ell\, \gamma_\mu \frac{1}{\not p_\ell+\not q}(\not p_\ell+\not q)(1-\gamma_5)v_\nu$$

$$= -ie f_B\, \bar u_\ell\, \not\epsilon\,(1-\gamma_5)v_\nu \qquad (2)$$

$$\left(= -ie f_B\, \bar u_\ell\, \Gamma_\mu v_\nu\, \epsilon^\mu\right).$$

See the $\pi \to \ell\nu\gamma$ case in the paper; note that besides not being gauge invariant, there is no helicity suppression

This has the typical structure of a contact, gauge non-invariant term. We expect that photon emission from the initial B meson contains a contact term which cancels the above term.

To see this explicitly, we use a generic covariant decomposition of the hadronic matrix element

$$T_{\mu S}^{(B)}(p,q) = i\int d^4x\, e^{iq\cdot x} \langle 0|T\{j_\mu^{em}(x)\,\bar u\,\Gamma_S\, b(0)\}|B^-(p+q)\rangle$$

in two independent 4-momenta p and q:

$$T_{\mu S}^{(B)}(p,q) = g_{\mu S}\, a + p_\mu q_S\, b + q_\mu p_S\, c + p_\mu p_S\, d + q_\mu q_S\, e + \epsilon_{S\mu\lambda\sigma}\, p^\lambda q^\sigma F_V^{(B)},$$

where a, b, c, d, e and $F_V^{(B)}$ are invariant amplitudes

$\sqrt{?}$

Why use only p and not p_ℓ, p ($p = p_\ell + p_\nu$) as independent momenta in the decomposition? → Strong int. knows nothing about ind. lepton moment a; only sees the mom. trans-ferred away from the flav. ...

Applying the (standard) e.m. Ward identity to the matrix element and using the conservation of the e.m. current (where there is an additional contribution due to the differentiation of the θ-function in the T-product) yields

∇? $\checkmark$

Where exactly is this add. contribution coming from and why does it look like this? Really from differentiating the θ-function? Also for e.m. piece do not have $q^\mu T^{(e.m.)}_{\mu s}$ → yes, from diff. the θ-fct in the time-ordered product; see also notes on "Ward identity in $B \to \ell \bar{\nu}_\ell \gamma$"

$$q^\mu T^{(B)}_{\mu s} = i (p+q)_s f_B .$$

Using this on the decomposition from before, we have

$$q^\mu T^{(B)}_{\mu s} = q_s a + (p\cdot q) q_s b + \overbrace{q^2 p_s c}^{=0} + (p\cdot q) p_s d + \overbrace{q^2 q_s e}^{=0}$$

$$+ \underbrace{\epsilon_{s\mu\lambda\sigma} q^\mu p^\lambda q^\sigma F_V^{(B)}}_{=0}$$

$$= q_s a + (p\cdot q) q_s b + (p\cdot q) p_s d$$

$$\overset{!}{=} i f_B (p+q)_s .$$

Comparing the coefficients at independent 4-momenta, one finds

$$q_s (a + (p\cdot q) b) \overset{!}{=} q_s i f_B \rightsquigarrow a + (p\cdot q) b = i f_B ,$$

$$p_s (p\cdot q) d \overset{!}{=} p_s i f_B \rightsquigarrow (p\cdot q) d = i f_B .$$

While the first relation connects the unknown amplitudes a and b, the second fixes the amplitude d.

∇? $\checkmark$

Why $F_V^{(B)}$ (vector) for the term with ϵ-tensor? Because B is a pseudoscalar, so we need ϵ-tensor for non-parity violation? → yes, and the other (axial) parts are only existent due to the weak current with $(1-\gamma_5)$; for $\pi^0 \to \gamma^* \gamma^*$ e.g. only ϵ-tensor part (coupling to two vector currents)

Hence, we can rewrite $T^{(B)}_{\mu s}$ according to

$$T^{(B)}_{\mu s}(p,q) = g_{\mu s} [i f_B - (p\cdot q) b] + p_\mu q_s b + q_\mu p_s c + p_\mu p_s \frac{i f_B}{(p\cdot q)}$$

$$+ q_\mu q_s e + \epsilon_{s\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(B)}$$

$$= -b\left[g_{\mu s}(p\cdot q) - p_\mu q_s\right] + i f_B g_{\mu s} + q_\mu p_s c + i \frac{p_\mu p_s}{(p\cdot q)} f_B$$

$$+ q_\mu q_s e + \varepsilon_{s\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(B)} \quad (*)$$

It is to be noted that upon contraction with q^μ, all terms except for the underlined trivially vanish. We can then furthermore rewrite (restoring the $-p_\mu q_s$ term).

$$T^{(B)}_{\mu s}(p,q) = \left[g_{\mu s}(p\cdot q) - p_\mu q_s\right] i F_A^{(B)} + g_{\mu s}(p\cdot q)\alpha + p_\mu q_s \beta$$

$$+ q_\mu p_s c + i \frac{p_\mu p_s}{(p\cdot q)} f_B + q_\mu q_s e + \varepsilon_{s\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(B)},$$

$$q^\mu T^{(B)}_{\mu s}(p,q) = q_s (p\cdot q)\alpha + (p\cdot q) q_s \beta + i f_B p_s ,$$

So that $\alpha + \beta = i \frac{f_B}{(p\cdot q)}$ in order to fulfill

$$q^\mu T^{(B)}_{\mu s}(p,q) = i(p+q)_s f_B .$$

Here, the values of α and β are arbitrary and not fixed from the e.m. Ward identity.

It is to be observed that the terms proportional to $F_A^{(B)}$, c, $\underline{e}$, and $F_V^{(B)}$ are gauge invariant, while the term proportional to f_B disappears in the chiral limit after being multiplied with the leptonic current $\bar{u}_e \Gamma^s v_\nu$, i.e.

$$\bar{u}_e \Gamma^s v_\nu p_s = \bar{u}_e \gamma^s (1-\gamma_5) v_\nu p_s = \bar{u}_e \not{p} (1-\gamma_5) v_\nu \overset{p = p_e + p_\nu}{=} 0 .$$

The remaining contact-term part of $T^{(B)}_{\mu s}$ containing α and β is gauge non-invariant. Different choices of α, β merely reflect different choices of $F_A^{(B)}$ and allow us to rewrite $T^{(B)}_{\mu s}$ in many ways.

To see this more explicitly, let us look at $(*)$ once more. We can add terms that vanish after contraction with q^μ,

This brings back degrees of freedom which had already been removed?? Or why should $i f_B g_{\mu s}$ not be sufficient? (Although only one new degree of freedom). Is that because $q^\mu T^{(B)}_{\mu s}$ always allows to add terms that trivially vanish under this operation?

Mistake in the paper? Also the e-term vanishes / is gauge inv.?

$$T^{(B)}_{\mu\delta}(p,q) = -b\left[g_{\mu\delta}(p\cdot q) - p_\mu q_\delta\right] + i f_B g_{\mu\delta} + q_\mu p_\delta c$$

$$+ i\frac{p_\mu p_\delta}{(p\cdot q)} f_B + q_\mu q_\delta e + \epsilon_{\delta\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(B)}$$

$$+ \alpha\left[p_\mu \cancel{p_\delta}^{\,q} - g_{\mu\delta}(p\cdot q)\right] \quad (25.08.2021)$$

$$\alpha p_\mu \cancel{q_\beta}_{\,\delta}\ (25.08.2021) - \alpha g_{\mu\delta}(p\cdot q) + i f_B g_{\mu\delta}$$

$$= \alpha p_\mu q_\beta + g_{\mu\delta}\underbrace{\left[i f_B - \alpha(p\cdot q)\right]}_{=\beta(p\cdot q)}$$

$$- \alpha g_{\mu\delta}(p\cdot q) - \beta p_\mu q_\delta + \alpha g_{\mu\delta}(p\cdot q) + \beta p_\mu q_\delta$$

$$= (\beta-\alpha)\left[g_{\mu\delta}(p\cdot q) - p_\mu q_\delta\right] + \alpha g_{\mu\delta}(p\cdot q) + \beta p_\mu q_\delta$$

$$= \underbrace{(-b+\beta-\alpha)}_{=iF_A^{(B)}}\left[g_{\mu\delta}(p\cdot q) - p_\mu q_\delta\right] + \alpha g_{\mu\delta}(p\cdot q) + \beta p_\mu q_\delta$$

$$+ q_\mu p_\delta c + i\frac{p_\mu p_\delta}{(p\cdot q)} f_B + q_\mu q_\delta e + \epsilon_{\delta\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(B)}.$$

(To find this "trick", we looked at $[g_{\mu\delta}(p\cdot q) - p_\mu q_\delta]\, iF_A^{(B)}$ and constructed $iF_A^{(B)} = -b - \alpha + \beta$ to cancel the additional terms proportional to α, β, whereupon we further investigated the remaining terms and then performed the steps in reverse direction.)

Let us now set $\beta = 0$, resulting in $\left(\alpha + \cancel{\beta} = \frac{i f_B}{(p\cdot q)}\right)$

$$T^{(B)}_{\mu\delta}(p,q) = \left[g_{\mu\delta}(p\cdot q) - p_\mu q_\delta\right] iF_A^{(B)} + i g_{\mu\delta} f_B + q_\mu p_\delta c$$

$$+ i\frac{p_\mu p_\delta}{(p\cdot q)} f_B + q_\mu q_\delta e + \epsilon_{\delta\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(B)}. \qquad (3)$$

Substituting Eq. (3) together with Eq. (2) into Eq (1), we obtain

$$A(B^- \to \ell \bar{\nu}_\ell \gamma) = \frac{G_F}{\sqrt{2}} V_{ub} (ie) \epsilon^{\mu} \left[(\bar{u}_\ell \Gamma^{S} v_\nu) \frac{T^{(B)}_{\mu S}(p,q)}{i} \right]$$
$$+ \frac{G_F}{\sqrt{2}} V_{ub} \left[-ie f_B (\bar{u}_\ell \Gamma^S v_\nu) \epsilon_S \right]$$

$$= e \frac{G_F}{\sqrt{2}} V_{ub}\, \epsilon^{\mu} \Big\{ (\bar{u}_\ell \Gamma^S v_\nu) \Big[(g_{\mu S} (p\cdot q) - p_\mu q_S)\, i F_A^{(B)} + i g_{\mu S} f_B$$
$$+ q_\mu p_S C + i \frac{p_\mu p_S}{(p\cdot q)} f_B + q_\mu q_S c + \epsilon_{S\mu\lambda\sigma} p^\lambda q^\sigma F_V^{(B)} \Big] \Big\}$$
$$+ \frac{G_F}{\sqrt{2}} V_{ub} \left[-ie f_B (\bar{u}_\ell \Gamma^S v_\nu) \epsilon_S \right]$$

Dirac eq. for p_S and $q\cdot\epsilon = 0$

$$= e \frac{G_F}{\sqrt{2}} V_{ub} \Big\{ (\bar{u}_\ell \Gamma^S v_\nu) \Big[(\epsilon_S (p\cdot q) - (\epsilon\cdot p) q_S)\, i F_A^{(B)} + i \epsilon_S f_B$$
$$+ \epsilon_{S\mu\lambda\sigma} \epsilon^\mu p^\lambda q^\sigma F_V^{(B)} \Big] - i (\bar{u}_\ell \Gamma^S v_\nu) \epsilon_S f_B \Big\}$$

for the decay amplitude. the terms in the brackets correspond to the initial state photon and the remaining term is the only effect of the final-state emission, the contact term from Eq. (2). As expected, the contact terms on the RHS cancel and the remaining "structure dependent" amplitude is a combination of two gauge invariant form factors.

Had we not set $\beta = 0$, we would have found

$$A(B^- \to \ell \bar{\nu}_\ell \gamma) = e \frac{G_F}{\sqrt{2}} V_{ub} \Big\{ (\bar{u}_\ell \Gamma^S v_\nu) \Big[(\epsilon_S (p\cdot q) - (\epsilon\cdot p) q_S)\, i F_A^{(B)}$$
$$= i f_B \epsilon_S - \beta (p\cdot q) \epsilon_S$$
$$+ \alpha \epsilon_S (p\cdot q) + \beta (\epsilon\cdot p) q_S + \epsilon_{S\mu\lambda\sigma} \epsilon^\mu p^\lambda q^\sigma F_V^{(B)} \Big]$$
$$- i (\bar{u}_\ell \Gamma^S v_\nu) \epsilon_S f_B \Big\}$$

$$= e \frac{G_F}{\sqrt{2}} V_{ub} \Big\{ (\bar{u}_\ell \Gamma^S v_\nu) \Big[(\epsilon_S (p\cdot q) - (\epsilon\cdot p) q_S) \left[i F_A^{(B)} - \beta \right]$$
$$+ i f_B \epsilon_S + \epsilon_{S\mu\lambda\sigma} \epsilon^\mu p^\lambda q^\sigma F_V^{(B)} \Big] - i (\bar{u}_\ell \Gamma^S v_\nu) \epsilon_S f_B \Big\},$$

which is also gauge invariant.

∇ ?

Contact term(s) also cancel(s) here? In the paper it is said that $\beta = 0$ is preferred because the contact terms vanish?!