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Semileptonic B-meson decays: Bhattacha, Feldmann, Wick 1

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In these notes, we perform some of the calculations from the paper arXiv:1004.3249v4 by A. Bhattacha, T. Feldmann, M. Wick for the g , which we later plan to transfer to m . (see also the corresponding Mathematica NB)

Verify these matrix elements (with DTT?)

not sure if current conserved, i.e. $\partial_\mu j^\mu$ density holds

Why $\bar{B}(p)$, i.e. anti-B meson?

Note: phase convention for vector state differs from some others by a relative factor of i (comment from paper)

We start by copying the (needed) matrix elements from the paper. For transitions between a pseudoscalar B meson and a generic (light) pseudoscalar meson, the hadronic matrix elements are usually written in terms of three form factors, $f_0(q^2)$, $f_+(q^2)$, and $f_T(q^2)$, which depend on the momentum transfer $q^2 = (p-k)^2$:

$$\langle P(k) | \bar{q} \gamma_\mu b | \bar{B}(p) \rangle = (p_\mu + k_\mu - q_\mu \frac{m_B^2 - m_P^2}{q^2}) f_+(q^2) + \frac{m_B^2 - m_P^2}{q^2} q_\mu f_0(q^2),$$

$$\langle P(k) | \bar{q} \sigma_{\mu\nu} q^\nu b | \bar{B}(p) \rangle = \frac{i}{m_B + m_P} (q^2 (p+k)_\mu - (m_B^2 - m_P^2) q_\mu) f_T(q^2).$$

At zero momentum transfer, the additional relation $f_+(0) = f_0(0)$ holds.

Similarly, the matrix elements for the transition of a B-meson and a generic vector meson can be written in terms of the FFs $V(q^2)$, $A_{0-3}(q^2)$, $T_{1-3}(q^2)$, conventionally defined via

$$\langle V(k, \epsilon) | \bar{q} \gamma_\mu b | \bar{B}(p) \rangle = i \epsilon_{\mu\nu\alpha\beta} \epsilon^{\nu\alpha} p^\beta k^\sigma \frac{2V(q^2)}{m_B + m_V}$$

$$\langle V(k, \epsilon) | \bar{q} \gamma_\mu \gamma_5 b | \bar{B}(p) \rangle = - \epsilon_\mu^\nu(k) (m_B + m_V) A_1(q^2) + (p+k)_\mu (\epsilon^\nu(k) \cdot q) \frac{A_2(q^2)}{m_B + m_V} + q_\mu (\epsilon^\nu(k) \cdot q) \frac{2m_V}{q^2} (A_3(q^2) - A_0(q^2)),$$

where $A_0(0) = A_3(0)$. For transitions involving a tensor current, the matrix elements are characterized by the tensor FFs,

$$\langle V(k, \epsilon) | \bar{q} \sigma_{\mu\nu} q^\nu b | \bar{B}(p) \rangle = i \epsilon_{\mu\nu\alpha\beta} \epsilon^{\nu\alpha} p^\beta k^\sigma 2T_1(q^2),$$

$$\langle V(k, \epsilon) | \bar{q} \sigma_{\mu\nu} q^\nu \gamma_5 b | \bar{B}(p) \rangle = T_2(q^2) [\epsilon_\mu^\nu(k) (m_B^2 - m_V^2) - (\epsilon^\nu(k) \cdot q) (p+k)_\mu] + T_3(q^2) (\epsilon^\nu(k) \cdot q) [q_\mu - \frac{q^2}{m_B^2 - m_V^2} (2p - q)_\mu]$$

Where $T_1(0) = T_2(0)$. The equations of motion for the quarks imply the additional constraint

$$A_3(q^2) = \frac{m_B + m_V}{2m_V} A_1(q^2) - \frac{m_B - m_V}{2m_V} A_2(q^2),$$

So that the $B \rightarrow V$ transitions are characterized by seven independent FFs.

It is convenient to take certain linear combinations of these "standard definition" FFs, which are referred to as helicity amplitudes in the paper (I think this name is not a good choice because I learned / got to know helicity amplitudes via a different definition). For one, these diagonalize the unitarity relations we will use/write down in a next step (which are then used to derive dispersive bounds on certain FF parameterizations). Moreover, they have definite spin-parity quantum numbers, which turns out useful when considering the contribution of excited states. In addition, they have simple relations to the universal FFs appearing in the heavy-quark and/or large-energy limit and lead to simple expressions for the observables in $B \rightarrow L L^+ L^-$ decays in the naive factorization approximation.

To also put the contributions to the various correlation functions entering the dispersive bounds on an equal footing, the authors choose a particular normalization convention.

For the $B \rightarrow P$ vector FFs, they define

$$A_{V,\sigma}(q^2) = \sqrt{\frac{q^2}{\lambda(m_B^2, m_P^2, q^2)}} E_{\sigma}^{*\dagger}(q) \langle P(k) | q \not{x} b | \bar{B}(p) \rangle,$$

where $E_{\sigma}^{*\dagger}(q)$ are transverse ($\sigma = \pm$), longitudinal ($\sigma = 0$), or time-like ($\sigma = t$) polarization vectors, defined as follows in the paper

Semileptonic B-meson decays: Bharuda, Feldman, Wick 2

$$E_{\pm}^{\mu}(q) = \mp \frac{1}{\sqrt{2}} (0, 1, \mp i, 0), \quad E_0^{\mu}(q) = \frac{1}{\sqrt{q^2}} (|\vec{q}|, 0, 0, -q^0),$$

$$E_{\pm}^{\mu}(q) = \frac{1}{\sqrt{q^2}} q^{\mu},$$

where $q^{\mu} = (q^0, 0, 0, -|\vec{q}|)$ is the corresponding four-momentum of the (virtual) vector state (all in rest frame of the B -meson in $B \rightarrow LX$, $L = P, V$ and X the said vector state). The three-momentum of the final-state meson points in positive z -direction, $k^{\mu} = (E, 0, 0, |\vec{k}|)$, where

$$|\vec{q}| = |\vec{k}| = \frac{\sqrt{\lambda(m_B^2, m_L^2, q^2)}}{2m_B}, \quad q^0 = m_B - E = \frac{m_B^2 - m_L^2 + q^2}{2m_B}$$

and the polarization vectors for an on-shell meson are given by

$$E_{\pm}^{\mu}(k) = \mp \frac{1}{\sqrt{2}} (0, 1, \pm i, 0), \quad E_0^{\mu}(k) = \frac{1}{m_L} (|\vec{k}|, 0, 0, E).$$

We also define the linear combinations

$$E_1^{\mu}(q) = \frac{E_{-}^{\mu}(q) - E_{+}^{\mu}(q)}{\sqrt{2}} = (0, 1, 0, 0)$$

$$E_2^{\mu}(q) = \frac{E_{-}^{\mu}(q) + E_{+}^{\mu}(q)}{\sqrt{2}} = (0, 0, i, 0)$$

This leads to

$$U_{T,0}(q^2) = f_{+}(q^2), \quad U_{T,\pm}(q^2) = \frac{m_B^2 - m_P^2}{\sqrt{\lambda(m_B^2, m_P^2, q^2)}} f_0(q^2),$$

$$\text{while } U_{V,\pm}(q^2) = 0.$$

For the $B \rightarrow P$ tensor FFs, they define

$$U_{T,0}(q^2) = (-i) \frac{1}{\sqrt{\lambda(m_B^2, m_P^2, q^2)}} E_0^{\mu\nu}(q) \langle P(k) | \bar{q} \sigma_{\mu\nu} q^{\nu} | B(q) \rangle.$$

The only non-vanishing FF is

$$U_{T,0}(q^2) = \frac{\sqrt{q^2}}{m_B + m_P} f_T(q^2).$$

usually defined
the long range
differently, see
my motivation notes.

This part is
extremely confusing
in the paper; in
particular, they
take about a
 k^{μ} (only) as
the on-shell
meson.

Note that
the linear
combinations
considered in the
main part are
different (extra
sign); for the
vector state, no
lin comb.
are considered.

$U_{T,0}(q^2)$ vanishes
as $q^2 \rightarrow 0$; however,
tensor current does
not contribute to phys.
processes at
anyway.

A similar analysis for the $B \rightarrow V$ vector and axial-vector FFs

yields

$$B_{V,0}(q^2) = \frac{\sqrt{q^2}}{\sqrt{\lambda(m_B^2, m_V^2, q^2)}} \sum_{\epsilon(k)} \epsilon_0^{*\mu}(q) \langle V(k, \epsilon(k)) | \bar{q} \gamma_\mu (1 - \gamma_5) b | B(p) \rangle,$$

with

$$B_{V,0}(q^2) = \frac{(m_B + m_V)^2 (m_B^2 - m_V^2 - q^2) A_1(q^2) - \lambda(m_B^2, m_V^2, q^2) A_2(q^2)}{2m_V \sqrt{\lambda(m_B^2, m_V^2, q^2)} (m_B + m_V)},$$

$$B_{V,t}(q^2) = A_0(q^2),$$

$$B_{V,1}(q^2) = -\frac{B_{V,-} - B_{V,+}}{\sqrt{2}} = \frac{\sqrt{2} \sqrt{q^2}}{m_B + m_V} V(q^2),$$

$$B_{V,2}(q^2) = -\frac{B_{V,-} + B_{V,+}}{\sqrt{2}} = \frac{\sqrt{2} \sqrt{q^2} (m_B + m_V)}{\sqrt{\lambda(m_B^2, m_V^2, q^2)}} A_1(q^2).$$

!

Note that the lin. combinations of the transv. pol. are different than what would result from using the pol. vectors with lin. combinations directly here.

At this, it is important to use a specific convention for the Levi-Civita tensor, in particular ^{the} one that does not match the one we usually use (d), e.g., for the $B \rightarrow \pi^+$ project. For the checked examples, this has no effect on the physical observables, however.

Finally, the $B \rightarrow V$ matrix elements with tensor currents are projected as

$$B_{T,0}(q^2) = \frac{1}{\sqrt{\lambda(m_B^2, m_V^2, q^2)}} \sum_{\epsilon(k)} \epsilon_0^{*\mu}(q) \langle V(k, \epsilon(k)) | \bar{q} \sigma_{\mu\nu} q^\nu (1 + \gamma_5) b | B(p) \rangle,$$

giving rise to the FFs

$$B_{T,0}(q^2) = \frac{\sqrt{q^2} (m_B^2 + 3m_V^2 - q^2)}{2m_V \sqrt{\lambda(m_B^2, m_V^2, q^2)}} T_2(q^2) - \frac{\sqrt{q^2} \sqrt{\lambda(m_B^2, m_V^2, q^2)}}{2m_V (m_B^2 - m_V^2)} T_3(q^2),$$

$$B_{T,1}(q^2) = -\frac{B_{T,-} - B_{T,+}}{\sqrt{2}} = \sqrt{2} T_1(q^2),$$

$$B_{T,2}(q^2) = -\frac{B_{T,-} + B_{T,+}}{\sqrt{2}} = \frac{\sqrt{2} (m_B^2 - m_V^2)}{\sqrt{\lambda(m_B^2, m_V^2, q^2)}} T_2(q^2).$$

Semileptonic B-meson decays: Branco, Feldmann, Wise 3

The FFs describe the process $B \rightarrow L$ with $L = P, V$ in the decay region $0 < q^2 < t_- = (m_B - m_L)^2$. Using crossing symmetry, they can also describe the process in the pair-production region $q^2 > t_+ = (m_B + m_L)^2$. This can be exploited to obtain a bound on parameters describing the FFs.

The crucial observation of the idea of dispersive bounds is the possibility to evaluate the correlator of two flavor-changing currents,

$$\Pi_{\mu\nu}^X(q^2) = i \int d^4x e^{iqx} \langle 0 | T j_\mu^X(x) j_\nu^{X\dagger}(0) | 0 \rangle,$$

either by an OPE or by unitarity considerations. The relevant currents are defined as

$$j_\mu^V = \bar{q} \gamma_\mu b,$$

$$j_\mu^{V-A} = \bar{q} \gamma_\mu (1 - \gamma_5) b,$$

$$j_\mu^T = \bar{q} \sigma_{\mu\alpha} q^\alpha b,$$

$$j_\mu^{T+A} = \bar{q} \sigma_{\mu\alpha} q^\alpha (1 + \gamma_5) b.$$

(Note that for phenomenological applications, we are only interested in the currents j_μ^{T+A} .)

Furthermore, we introduce longitudinal and transverse helicity projection,

$$P_L^{\mu\nu}(q^2) = \frac{q^\mu q^\nu}{q^2}, \quad P_T^{\mu\nu}(q^2) = \frac{1}{D-1} \left(\frac{q^\mu q^\nu}{q^2} - g^{\mu\nu} \right), \quad D=4,$$

which allow us to rewrite the correlation functions in terms of Lorentz scalar quantities,

$$\Pi_I^X(q^2) = P_I^{\mu\nu}(q^2) \Pi_{\mu\nu}^X(q^2), \quad I = L, T.$$

As $\Pi_I^X(q^2)$ is an analytic function, it satisfies the subtracted dispersion relation

$$X_I^X(u) = \frac{1}{n!} \frac{d^n \Pi_I^X(q^2)}{dq^{2n}} \Big|_{q^2=0} = \frac{1}{n} \int_0^\infty dt \frac{\ln \Pi_I^X(t)}{(t-q^2)^{n+1}} \Big|_{q^2=0}$$

Actually depend on q^2 , not only on q^2 !!

Why do they evaluate this eq. at $q^2=0$?
OPE result calculated at $q^2=0$!!

Where the number of subtractions n is chosen to render the resulting function $\chi_{\Gamma}^{\times}(\omega)$ finite, this can be shown to hold for any specific value of n from the standard form of subtracted dispersion relations.

Should probably write a proof by induction (thesis?)

Unitarity allows one to express $\text{Im} T_{\Gamma}^{\times}(q^2)$ as the positive-definite sum over all hadronic states Γ with the allowed quantum numbers as follows:

$$\text{Im} T_{\Gamma}^{\times}(q^2) = \frac{1}{2} \sum_{\Gamma} \int d\Omega_{\Gamma} (2\pi)^4 \delta^{(4)}(q-p_{\Gamma}) P_{\Gamma}^{\mu\nu} \langle 0 | j_{\Gamma}^{\times}(\omega) | \Gamma \rangle \langle \Gamma | j_{\Gamma}^{\times\dagger}(\omega) | 0 \rangle,$$

where p_{Γ} is the total momentum of the final state and $d\Omega_{\Gamma}$ contains the appropriate phase-space weighting,

$$d\Omega_{\Gamma} = \pi \frac{d^3 p_j}{(2\pi)^3 2p_j^0} = \pi \frac{d^4 p_j}{(2\pi)^4} (2\pi) \delta(p_j^2 - m_j^2) \theta(p_j^0).$$

For a particular choice of intermediate state, $\Gamma = BL$, we have

$$d\Omega_{BL} = \frac{d^3 p_B d^3 p_L}{(2\pi)^3 2p_B^0 (2\pi)^3 2p_L^0}$$

and we define

$$\text{Im} T_{I, BL}^{\times}(q^2) = \frac{\eta}{2} \underbrace{\int \frac{d^3 p_B d^3 p_L}{(2\pi)^3 2p_B^0 (2\pi)^3 2p_L^0}}_{\text{two-body phase space}} (2\pi)^4 \delta^{(4)}(q - (p_B + p_L)) P_{I}^{\mu\nu} \langle 0 | j_{\Gamma}^{\times}(\omega) | BL \rangle \langle BL | j_{\Gamma}^{\times\dagger}(\omega) | 0 \rangle,$$

What about one-particle int. states? Or only consider BL here because we want to learn st. about B → L FF? we have to be considered in principle. See Damski paper.

where η is an isospin-degeneracy factor for a given channel. Here, the two-body phase space evaluates to

$$\int \frac{d^3 p_B d^3 p_L}{(2\pi)^3 2p_B^0 (2\pi)^3 2p_L^0} = \frac{1}{4\pi} \frac{|\vec{p}_{\text{cm}}|}{E_{\text{cm}}}, \quad \text{where } |\vec{p}_{\text{cm}}| \text{ is the three-mom. of the B- or L-meson in their CMS and } E_{\text{cm}} = \sqrt{q^2} = \sqrt{s} \text{ their total energy}$$

What is this isospin-degeneracy factor? For S_1 apparently $3/2$, $K \cong 2V$, $\phi \cong 1$, see paper → different channels that contribute; for kaons have ϕ K⁰ B⁰ S⁰ → $\frac{1}{2} \sqrt{2} \frac{1}{\sqrt{2}}$ Note that it's probably more transparent to calculate the matrix elements before performing the integration (have to do this in CMS, i.e. frame of q^2)

Hence

$$\text{Im} T_{I, BC}^{\times}(q^2) = \frac{\eta}{2} \frac{1}{4\pi} \frac{\sqrt{\lambda(m_B^2, m_L^2, q^2)}}{2q^2} P_{I}^{\mu\nu} \langle 0 | j_{\Gamma}^{\times}(\omega) | BL \rangle \langle BL | j_{\Gamma}^{\times\dagger}(\omega) | 0 \rangle$$

$$= \eta \frac{\sqrt{\lambda(m_B^2, m_L^2, q^2)}}{16\pi q^2} P_{I}^{\mu\nu} \langle 0 | j_{\Gamma}^{\times}(\omega) | BL \rangle \langle BL | j_{\Gamma}^{\times\dagger}(\omega) | 0 \rangle.$$

The factor of $1/2$ went missing in the paper here but reappeared in the next line (20)

frame of q^2

Semileptonic B-meson decays: Bhattacharya, Feldmann, Wick 4

Rewriting the standard basis of form factors in terms of the ones from the helicity amplitudes, we find

$$P_T^{\mu\nu} \langle P | j_\mu^\nu | B \rangle \langle B | j_\nu^{\mu\dagger} | P \rangle = \frac{\lambda(m_B^2, m_P^2, q^2)}{3q^2} |A_{T,10}|^2,$$

$$P_L^{\mu\nu} \langle P | j_\mu^\nu | B \rangle \langle B | j_\nu^{\mu\dagger} | P \rangle = \frac{\lambda(m_B^2, m_P^2, q^2)}{3q^2} |A_{T,1\pm}|^2,$$

$$P_T^{\mu\nu} \langle P | j_\mu^T | B \rangle \langle B | j_\nu^{\mu\dagger} | P \rangle = \frac{\lambda(m_B^2, m_P^2, q^2)}{3} |A_{T,10}|^2.$$

for $B \rightarrow P$ (whereas $P_L^{\mu\nu}$ on the tensor ones vanish),

$$P_T^{\mu\nu} \langle V | j_\mu^{\nu-A} | B \rangle \langle B | j_\nu^{\mu\dagger \nu-A} | V \rangle = \frac{\lambda(m_B^2, m_V^2, q^2)}{3q^2} \sum_{i=0}^2 |B_{V,i}|^2,$$

$$P_L^{\mu\nu} \langle V | j_\mu^{\nu-A} | B \rangle \langle B | j_\nu^{\mu\dagger \nu-A} | V \rangle = \frac{\lambda(m_B^2, m_V^2, q^2)}{q^2} |B_{V,\pm}|^2,$$

$$P_T^{\mu\nu} \langle V | j_\mu^{T+A} | B \rangle \langle B | j_\nu^{\mu\dagger T+A} | V \rangle = \frac{\lambda(m_B^2, m_V^2, q^2)}{3} \sum_{i=0}^2 |B_{V,i}|^2$$

for $B \rightarrow V$ (where $P_L^{\mu\nu}$ or $T+A$ again vanishes).

We can thus write

$$\text{Im} \Pi_{F,SL}^x(q^2) = \eta \frac{\lambda(m_B^2, m_L^2, q^2)^{3/2}}{48\pi q^4} |A_{F,SL}^x|^2,$$

where

$$|A_{T,10}^V|^2 = |A_{V,10}|^2, \quad |A_{L,1\pm}^V|^2 = 3 |A_{V,\pm}|^2, \quad |A_{T,10}^T|^2 = q^2 |A_{T,10}|^2.$$

for $B \rightarrow P$ and

$$|A_{T,1\pm}^{V-A}|^2 = \sum_{i=0}^2 |B_{V,i}|^2, \quad |A_{L,1\pm}^{V-A}|^2 = 3 |B_{V,\pm}|^2,$$

$$|A_{T,10}^{T+A}|^2 = q^2 \sum_{i=0}^2 |B_{V,i}|^2$$

for $B \rightarrow V$.

Since $\text{Im} \Pi_{F,SL}^x(H) \geq \text{Im} \Pi_{F,SL}^x(L)$, we now find

$$\chi_I^x(n) = \frac{1}{\pi} \int_0^\infty dt \frac{\ln \pi_{\pm}^x(t)}{(t - q^2)^{n+1}} \Big|_{q^2=0} \geq \frac{1}{\pi} \int_{t_+}^\infty dt \frac{\ln \pi_{\pm}^x(t)}{(t - q^2)^{n+1}} \Big|_{q^2=0}$$

$$= \frac{1}{\pi} \int_{t_+}^\infty dt \eta \frac{\lambda(m_1^2, m_2^2, t)^{3/2}}{48\pi t^2} \frac{|\ln \pi_{\pm}^x(t)|^2}{(t - q^2)^{n+1}} \Big|_{q^2=0} = \frac{1}{\pi} \int_{t_+}^\infty dt \eta \frac{\lambda(m_1^2, m_2^2, t)^{3/2}}{48\pi t^{n+3}} |\chi_I^x(t)|^2$$

Here, $\chi_I^x \equiv \chi_{I, \text{free}}^x$ is calculated from the OPE.

Next, we want to use the conformal mapping

$$Z(\lambda) \equiv Z(t, t_0) = \frac{\sqrt{t_+ - t'} - \sqrt{t_+ - t_0'}}{\sqrt{t_+ - t'} + \sqrt{t_+ - t_0'}} \quad , \quad 0 \leq t_0 < t_+$$

free parameter, can be optimized

to reduce maximum value of $|Z(\lambda)|$

FF well described by SE (see below) truncated after the second term proportional to $Z(\lambda)$.

in physical FF range,

$$t_0^{\text{opt}} = t_+ \left(1 - \sqrt{1 - \frac{t_-}{t_+}}\right)$$

to map the t -plane cut from $[t_+, \infty)$ onto the unit disk. At this, we extended the FFs defined in the physical range $q^2 \in [0, t_-]$ to analytic functions throughout the complex t -plane except for along the branch cut starting at the threshold for the production of real BP/BV pairs, $q^2 \geq t_+$. If low-lying resonances are present below t_+ (with appropriate quantum numbers and mass m_R), they are accounted for by the so-called Blaschke factor $B(\lambda)$.

The variable $Z(\lambda)$ is found to be an excellent expansion parameter for the FFs. With an appropriately chosen normalization function $\phi_f(t)$, one obtains simple dispersive bounds on the coefficients of the series expansion

$$f(\lambda) = \frac{1}{B(\lambda)\phi_f(\lambda)} \sum_k a_k Z^k(t).$$

Since $|Z(\lambda)| = 1$ in the pair production region, $t \geq t_+$, the Blaschke factor is chosen as $B(\lambda) = Z(t, m_R^2)$ (which is readily obtained from

$$B(t) = \frac{Z(t) - Z(m_R^2)}{1 - Z(m_R^2)^* Z(t)}.$$

Note that in the beginning of the paper, it is stated that the range of these low-lying resonances is $t_- < m_R < t_+$.

Semileptonic B-meson decays: Bhambhani, Feldmann, Wick 5

Using that

$$\frac{1}{2\pi i} \int_{\mathcal{D}} \frac{dz}{z} |f(z(t))|^2 = \frac{1}{\pi} \int_{t_+}^{\infty} dt \left| \frac{dz(t)}{dt} \right| |f(z(t))|^2$$

$$f = \phi_{\pm}^{\pm} \Rightarrow \frac{1}{2\pi i} \int_{\mathcal{D}} \frac{dz}{z} |\phi_{\pm}^{\pm}(z) A_{\pm}^{\pm}(z)|^2 = \frac{1}{\pi} \int_{t_+}^{\infty} dt \left| \frac{\sqrt{t-t_0}}{\sqrt{t-t_+}(\sqrt{t-t_+} + \sqrt{t-t_0})^2} \right| \times |\phi_{\pm}^{\pm}(t) A_{\pm}^{\pm}(t)|^2$$

$$\stackrel{|\sqrt{t-t_+} + \sqrt{t-t_0}|^2 = t-t_0}{=} \frac{1}{\pi} \int_{t_+}^{\infty} dt \frac{\sqrt{t-t_0}}{\sqrt{t-t_+}(t-t_0)} |\phi_{\pm}^{\pm}(t) A_{\pm}^{\pm}(t)|^2$$

We find when making to

$$\frac{1}{\pi} \int_{t_+}^{\infty} dt \eta \frac{\lambda(m_B^2, m_c^2, t)^{3/2}}{48\pi t^{n+3} \chi_{\pm}^{\pm}(t)} |A_{\pm}^{\pm}(t)|^2 \leq 1$$

that (on the cut)

$$|\phi_{\pm}^{\pm}(t)|^2 = \eta \frac{\lambda(m_B^2, m_c^2, t)^{3/2}}{48\pi t^{n+3} \chi_{\pm}^{\pm}(t)} \frac{\sqrt{t-t_+}(t-t_0)}{\sqrt{t-t_0}}$$

$$\left. \begin{aligned} t_{\pm} &= (m_B \pm m_c)^2 \Rightarrow m_B = \frac{\sqrt{t_+} + \sqrt{t_-}}{2}, m_c = \frac{\sqrt{t_+} - \sqrt{t_-}}{2} \\ \lambda(m_B^2, m_c^2, t) &= (t-t_+)(t-t_-) \end{aligned} \right\}$$

$$= \frac{\eta}{48\pi \chi_{\pm}^{\pm}(t)} \frac{(t-t_+)^2}{(t-t_0)^{3/2}} \frac{(t-t_-)^{3/2}}{t^{n+2}} \frac{t-t_0}{t} \quad (*)$$

Why do they split t^{n+2} into $t^{n+1} \cdot t$?

In the paper it's stated that these factors take into account the "non-conventional" normalization of our FFs.

Recall now that from the change of basis in the FFs, some of them carry the prefactor $\sqrt{q^2}$ and/or $\frac{1}{\sqrt{\lambda(m_B^2, m_c^2, q^2)}}$. The corresponding Blaschke factors are given by $\sqrt{-z(t_0)}$ and $\sqrt{z(t_+, t_-)}$, respectively.

Respecting the additional poles from subthreshold resonances of mass m_{c_i} by the Blaschke factor $B(t) = \prod_i z(t, m_{c_i}^2)$, where $|B(t)| = 1$ in the

Why $\sqrt{-z(t_0)}$ for $t_0 < 0$ and $\sqrt{z(t_+, t_-)}$ for $t_+ > 0$ and $t_- > 0$ and $z(t, t_0) < 0$, so that one has to choose $\sqrt{-z(t_0)}$ with a sign.

pair production region, we have $\chi = (t_+)(t_-)$ we have that, e.g. for $t_- < t < t_+$, $\chi < 0$. Then, $z(t, t_-) < 0$, which exactly cancels the sign and leads to an analytic $\chi(t)$. For $t \leq t_+$ (where $z(t, t_+) \geq 0$ and $z(t, t_-) > 0$), $\chi > 0$ and $z(t, t_+) > 0$ and $z(t, t_-) > 0$.

$$A_{\pm}^x(t) = \frac{\sqrt{-z(t_0)}^m \sqrt{z(t, t_0)}^{n-l}}{B(t) \Phi_{\pm}^x(t)} \sum_{k=0}^{\infty} \alpha_k z^k, \quad (***)$$

with $m=1$ e.g. for $U_{\pm,0}$ and $l=1$ for $U_{\pm,1}$.

The function $\Phi_{\pm}^x(t)$ has to be constructed in such a way that its absolute value satisfies (**), while (***) retains the analytical properties of the FF. This can be achieved by replacing potential poles and cuts in $\sqrt{|\Phi_{\pm}^x(t)|^2}$ according to

$$\frac{1}{t-x} \rightarrow \frac{-z(t,x)}{t-x},$$

which is allowed because $|z(t,x)|=1$ in the pair production region.

We then find

$$\Phi_{\pm}^x(t) = \sqrt{\frac{\eta}{48\pi\alpha'|\ln|}} \frac{(t-t_0)}{(t_0-t_0)^{1/4}} \left(\frac{-z(t_0)}{t}\right)^{(n+3)/2} \left(\frac{t-t_0}{-z(t,t_0)}\right)^{1/2} \left(\frac{t-t_0}{-z(t,t_0)}\right)^{3/4}$$

When inserting (***) into the dispersive band (integral equation), we find

$$\frac{1}{2\pi i} \int_{\partial D} \frac{dz}{z} \underbrace{|\Phi_{\pm}^x(z) A_{\pm}^x(z)|^2}_{=1} \leq 1$$

$$= |B(t) \Phi_{\pm}^x(t) A_{\pm}^x(t) [-z(t_0)]^{m/2} [z(t,t_0)]^l|^2 = \left| \sum_{k=0}^{\infty} \alpha_k z^k \right|^2$$

$$\Leftrightarrow \frac{1}{2\pi i} \int_{\partial D} \frac{dz}{z} \left[\sum_k \alpha_k z^k \right] \left[\sum_l \alpha_l z^{*l} \right] = \frac{1}{2\pi i} \sum_k \sum_l \alpha_k \alpha_l^* \underbrace{\int_{\partial D} \frac{dz}{z} z^k z^{*l}}_{=2\pi i \delta_{kl}}$$

$$= \sum_k \alpha_k^2$$

For $B \rightarrow V$, an analogous parameterization as (***) is used for each individual FF in (recall)

$$|A_{\pm}^{V-A}|^2 = \sum_{i=0}^2 |B_{V,i}|^2,$$

$$|A_{\pm}^{V-A}|^2 = 3 |B_{V,1}|^2,$$

$$|A_{\pm}^{T+A}|^2 = 9 \sum_{i=0}^2 |B_{T,i}|^2.$$

Also, let for $B_{V,0}$, $m=1$ for $B_{V,1}$, $m=1, l=1$ for $B_{V,2}$, let for $B_{T,1}$, $m=1, l=1$ for $B_{T,0}$

? For cuts, $t-x$ is not in the denominator, right?

Why $\sqrt{(t-t_0)^2} = t-t_0$ and not t_0-t ? Same argument as for the other terms would lead to sign change here

Semileptonic B-meson decays: Bhambhani, Feldman, Wick 6

This leads to a bound on the sum of the corresponding coefficients. The paper considers $A_T^{r-A}(t)$ as an example:

$$B_{V_0}(t) = \frac{1}{B(t) \sqrt{z(t,t_0)} A_T^{r-A}(t)} \sum_{k=0}^{K-1} \beta_k^{(V_0)} z^k,$$

$$B_{V_2}(t) = \frac{\sqrt{-z(t,t_0)}}{B(t) A_T^{r-A}(t)} \sum_{k=0}^{K-1} \beta_k^{(V_2)} z^k,$$

$$B_{V_2}(t) = \frac{\sqrt{-z(t,t_0)}}{B(t) \sqrt{z(t,t_0)} A_T^{r-A}(t)} \sum_{k=0}^{K-1} \beta_k^{(V_2)} z^k.$$

Then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{dz}{z} \left| \underbrace{\phi_T^x(z) A_T^x(z)} \right|^2 \leq 1$$
$$= \sum_{i=0}^2 \left| \sum_{k=0}^{K-1} \beta_k^{(V_i)} z^k \right|^2$$

$$\Leftrightarrow 1 \geq \frac{1}{2\pi i} \int_{\partial D} \frac{dz}{z} \sum_{i=0}^2 \left[\sum_k \beta_k^{(V_i)} z^k \right] \left[\sum_l \beta_l^{(V_i)} z^{*l} \right]$$

$$= \frac{1}{2\pi i} \sum_{i=0}^2 \sum_k \sum_l \beta_k^{(V_i)} \beta_l^{(V_i)} \underbrace{\int_{\partial D} \frac{dz}{z} z^k z^{*l}}_{= \delta_{kl}}$$

$$= \sum_{i=0}^2 \sum_k \left[\beta_k^{(V_i)} \right]^2$$