

## Disclaimer

The notes at hand were written during my research period as a PhD student at the University of Bonn. They contain auxiliary calculations to and comments on publications by other authors, which are subject to definite conditions of use; see also the respective article(s) on <https://arxiv.org/> linked on the following website. For more information and all my material, check: <https://www.physics-and-stuff.com/>

**I raise no claim to correctness and completeness of the given material!**

Note that the license of the respective article(s) on <https://arxiv.org/> applies. Do not cite these notes but the original authors and the published article as given on <https://www.physics-and-stuff.com/>.

# Polarization Sums of Spin-1, Spin-2, and Spin-3 Particles (Massive) 1

26.04.2021

We want to derive the polarization sum for massive spin-3 particles as given in <1203.2501> (Appendix) (Nickig et al.) in the following. For completeness and clarification, we also derive the polarization sums for massive spin-1 and spin-2 particles.

Let us start with the most simple case, spin-1; an ansatz for the polarization sum

$$X_{\mu\nu} = \sum_{\text{pol.}} \epsilon_{\mu}(p, \lambda) \epsilon_{\nu}^*(p, \lambda)$$

the most general

reads

$$X_{\mu\nu} = A g_{\mu\nu} + B p_{\mu} p_{\nu}$$

Lorentz-covariant

Since these are the only available building blocks, this tensor has to fulfill the properties

$$p^{\mu} X_{\mu\nu} = 0 = X_{\mu\nu} p^{\nu}, \quad g^{\mu\nu} X_{\mu\nu} = -3,$$

Since  $p \cdot \epsilon = 0$  and  $\epsilon_{\mu}(p, \lambda) \epsilon^{\mu}(p, \lambda') = -\delta_{\lambda\lambda'}$ ,

Using Mathematica, we solve the resulting system of equations and find  $A = -1$ ,  $B = \frac{1}{M^2}$ , where  $M$  is the mass of the spin-1 particle. Here, we used on-shell kinematics,  $p^2 = M^2$ . The polarization sum thus reads

$$X_{\mu\nu} = -g_{\mu\nu} + \frac{p_{\mu} p_{\nu}}{M^2}.$$

Why on-shell kinematics,  $p^2 = M^2$ ? Because it's a pol. sum, so as e.g. for Dirac spin sums, which are always considered on-shell? Why are they?

For spin=2, the polarization vectors carry an additional index,

$$P_{2\mu_1\mu_2}^{\nu_1\nu_2} = \sum_{p_1} E_{\mu_1\mu_2}(p_1, X) E^{\nu_1\nu_2}(p_1, X),$$

which is totally symmetric in  $\mu_1 \leftrightarrow \mu_2$  and  $\nu_1 \leftrightarrow \nu_2$  and where similar constraints to before need to hold,

$$P_{\mu_1}^{\nu_1} P_{2\mu_1\mu_2}^{\nu_1\nu_2} = 0 \quad (= P_{\mu_2}^{\nu_2} P_{2\mu_1\mu_2}^{\nu_1\nu_2}), \quad P_{\mu_1} P_{2\mu_1\mu_2}^{\nu_1\nu_2} = 0 \quad (= P_{\nu_2} P_{2\mu_1\mu_2}^{\nu_1\nu_2}),$$

$$g_{\mu_1}^{\nu_1} g_{\mu_2}^{\nu_2} P_{2\mu_1\mu_2}^{\nu_1\nu_2} = -5,$$

Why totally sym? Arises from the field's properties?! But why does the field have to be such?

with the additional constraint that (traces, found on the internet)

$$g_{\mu_1}^{\nu_1} P_{2\mu_1\mu_2}^{\nu_1\nu_2} = 0 = g_{\nu_1\nu_2} P_{2\mu_1\mu_2}^{\nu_1\nu_2} \implies g_{\mu_1}^{\nu_1} g_{\nu_1\nu_2} P_{2\mu_1\mu_2}^{\nu_1\nu_2} = 0$$

Where is this additional constraint coming from? (No information seems to be lost when combining these into one property)

Additionally to the rank-2 tensor  $X_{\mu\nu} = -g_{\mu\nu} + \frac{P_{\mu\nu}}{M^2}$ , the rank-4 tensor  $P_{2\mu_1\mu_2}^{\nu_1\nu_2}$  can consist of  $\tilde{X}_{\mu\nu} = -g_{\mu\nu} - \frac{P_{\mu\nu}}{M^2}$  (so that  $X_{\mu\nu}$  and  $\tilde{X}_{\mu\nu}$  can be combined into all possible rank-2 tensor). The most general ansatz thus reads

$$P_{2\mu_1\mu_2}^{\nu_1\nu_2} = A_1 (X_{\mu_1}^{\nu_1} X_{\mu_2}^{\nu_2} + X_{\mu_2}^{\nu_1} X_{\mu_1}^{\nu_2}) + A_2 X_{\mu_1\mu_2} X^{\nu_1\nu_2} + A_3 (\tilde{X}_{\mu_1}^{\nu_1} X_{\mu_2}^{\nu_2} + \tilde{X}_{\mu_2}^{\nu_1} X_{\mu_1}^{\nu_2}) + A_4 \tilde{X}_{\mu_1\mu_2} X^{\nu_1\nu_2} + A_5 (X_{\mu_1}^{\nu_1} \tilde{X}_{\mu_2}^{\nu_2} + X_{\mu_2}^{\nu_1} \tilde{X}_{\mu_1}^{\nu_2}) + A_6 X_{\mu_1\mu_2} \tilde{X}^{\nu_1\nu_2} + A_7 (\tilde{X}_{\mu_1}^{\nu_1} \tilde{X}_{\mu_2}^{\nu_2} + \tilde{X}_{\mu_2}^{\nu_1} \tilde{X}_{\mu_1}^{\nu_2}) + A_8 \tilde{X}_{\mu_1\mu_2} \tilde{X}^{\nu_1\nu_2},$$

$E_{\mu_1\mu_2}^{\nu_1\nu_2}$  has the wrong symmetry

which is already properly symmetrized.

Imposing the constraints in Mathematica and solving the resulting system of equations yields

$$A_2 = -2A_1 - \frac{2}{3}, \quad A_3 = A_1 + \frac{1}{2}, \quad A_4 = -2A_1 - 1, \quad A_5 = -A_1 - \frac{1}{2}, \\ A_6 = 2A_1 + 1, \quad A_7 = -A_1 - \frac{1}{2}, \quad A_8 = 2A_1 + 1.$$

A clever choice is  $A_1 = -\frac{1}{2}$ , resulting in  $A_2 = \frac{1}{3}, A_i = 0, i = 3, \dots, 8,$

i.e. 
$$P_{2\mu_1\mu_2}^{\nu_1\nu_2} = -\frac{1}{2} (X_{\mu_1}^{\nu_1} X_{\mu_2}^{\nu_2} + X_{\mu_2}^{\nu_1} X_{\mu_1}^{\nu_2}) + \frac{1}{3} X_{\mu_1\mu_2} X^{\nu_1\nu_2}.$$

Why is there freedom left to choose a constant? Missing a constraint?  $P_{\mu_1\mu_2}^{\nu_1\nu_2} = -5$  also not bring new constraint is also a constraint comp

Note that leaving out the  $\tilde{X}$ -terms in the decomposition yields

this polarization sum resembles the graviton

# Polarization Sums of Spin -1, Spin -2, and Spin -3 Particles (Massive) 2

For spin = 3, we will now be guided by the considerations from the paper; note that there, the particle is fixed to the  $g_3$  meson. Scattering resonance is described by a totally symmetric rank-3 tensor field  $S_{\mu\nu\lambda} = S_{\mu\nu\lambda}^a T^a$  ( $T=1$ ), which is subject to the constraints

$$\partial^\mu S_{\mu\nu\lambda} = 0, \quad g^{\mu\nu} S_{\mu\nu\lambda} = 0.$$

We define the polarization sum as per

$$P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3} = \sum_{\text{pol.}} \epsilon_{\mu\nu\rho\sigma}(p, k) \epsilon^{\nu_1\nu_2\nu_3}(p, l),$$

which is then subject to the constraints

$$p^\mu P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3} = 0 = P_{\nu_i} P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3}, \quad g^{\mu\nu} P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3} = 0 = g_{\nu_i\nu_j} P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3}$$

$\implies g^{\mu\nu} g_{\nu_1\nu_2} P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3} = 0$  / Sufficient to use this constraint for the combined property with  $i=1, j=2, k=1, l=2$ .

$$g_{\nu_1}^{\mu_1} g_{\nu_2}^{\mu_2} g_{\nu_3}^{\mu_3} P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3} = -7.$$

$\epsilon$ -tensor has wrong sym. property

The most general ansatz for the polarization sum then reads

$$P_{3\mu\nu\rho\sigma}^{\nu_1\nu_2\nu_3} = A_1 \sum_{\mu_1, \nu_1, \mu_2, \nu_2, \mu_3, \nu_3} X_{\mu_1}^{\nu_1} X_{\mu_2}^{\nu_2} X_{\mu_3}^{\nu_3} + A_2 \left\{ \sum_{\mu_1, \nu_1} X_{\mu_1\nu_1} X_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} + \sum_{\mu_1, \nu_1} X_{\mu_1\nu_1} X_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} + \sum_{\mu_1, \nu_1} X_{\mu_1\nu_1} X_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} \right\}$$

$$+ A_3 \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1}^{\nu_1} X_{\mu_2}^{\nu_2} X_{\mu_3}^{\nu_3} + A_4 \left\{ \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1\nu_1} X_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} + \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1\nu_1} X_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} + \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1\nu_1} X_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} \right\}$$

+ (second  $X \rightarrow \tilde{X}$  instead of first) + (third  $X \rightarrow \tilde{X}$  instead of first)

$$+ A_9 \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1}^{\nu_1} \tilde{X}_{\mu_2}^{\nu_2} X_{\mu_3}^{\nu_3} + A_{10} \left\{ \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1\nu_1} \tilde{X}_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} + \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1\nu_1} \tilde{X}_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} + \sum_{\mu_1, \nu_1} \tilde{X}_{\mu_1\nu_1} \tilde{X}_{\mu_2}^{\nu_1\nu_2} X_{\mu_3}^{\nu_3} \right\}$$

+ (second  $\tilde{X}$  remains  $X$ ) + (first  $\tilde{X}$  remains  $X$ )

$$+ A_{15} \sum_{P \in \mathcal{P}} \tilde{X}_{\mu_1}^{v_1} \tilde{X}_{\mu_2}^{v_2} \tilde{X}_{\mu_3}^{v_3} + A_{16} \left\{ \sum_{P \in \mathcal{P}} \tilde{X}_{\mu_1 \mu_2}^{v_1 v_2} \tilde{X}_{\mu_3}^{v_3} + \sum_{P \in \mathcal{P}} \tilde{X}_{\mu_1 \mu_3}^{v_1 v_3} \tilde{X}_{\mu_2}^{v_2} \right. \\ \left. + \sum_{P \in \mathcal{P}} \tilde{X}_{\mu_2 \mu_3}^{v_2 v_3} \tilde{X}_{\mu_1}^{v_1} \right\}.$$

Imposing the aforementioned constraints in Mathematica results in

$$A_1 = 1/6, \quad A_2 = -1/30, \quad A_i = 0, \quad i = 3, \dots, 16.$$

Hence,

$$\mathbb{P}_{3\mu_1\mu_2\mu_3}^{v_1v_2v_3} = \frac{1}{6} \sum_{P \in \mathcal{P}(v_1, v_2, v_3)} \left\{ X_{\mu_1}^{v_1} X_{\mu_2}^{v_2} X_{\mu_3}^{v_3} - \frac{1}{5} \left( X_{\mu_1 \mu_2}^{v_1 v_2} X_{\mu_3}^{v_3} + X_{\mu_1 \mu_3}^{v_1 v_3} X_{\mu_2}^{v_2} \right. \right. \\ \left. \left. + X_{\mu_2 \mu_3}^{v_2 v_3} X_{\mu_1}^{v_1} \right) \right\} \\ = \frac{1}{6} \sum_{P \in \mathcal{P}(v_1, v_2, v_3)} \left\{ X_{\mu_1}^{v_1} X_{\mu_2}^{v_2} X_{\mu_3}^{v_3} - \frac{1}{5} \left( X_{\mu_1 \mu_2}^{v_1 v_2} X_{\mu_3}^{v_3} + X_{\mu_1}^{v_1} X_{\mu_2 \mu_3}^{v_2 v_3} \right. \right. \\ \left. \left. + X_{\mu_1 \mu_3}^{v_1 v_3} X_{\mu_2}^{v_2} \right) \right\}$$