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Unitarity Relation for $V \rightarrow 3\pi (V \rightarrow \pi\pi) \dagger$

20.04.2021 In the following, we want to confirm some calculations from <1203.2504> (Niedlung et al), in particular Eq. (13) therein.

Schneider Dis. can be absorbed in det. of $\Gamma(s)$ or $1/2$ spin amplitude
 Factor of $1/2$ actually only in imaginary part but not disc!
 And where from factor of i ? Due to the fact that we "usually" have $M = \dots$ and then $2\pi i = 2\pi i - i$ from crossing on RHS?!
 Note that for the former question, factor needs to be absorbed somewhere, e.g. in $\Gamma(s)$.

We start from Eq. (12),

$$\text{disc } M(s, z_s) = \frac{i}{2} \int \frac{d^4 l}{(2\pi)^4} M(s, z_s') T^{*} (s, z_s'') \times [\pi] \delta(l^2 - m_\pi^2) [\pi] \delta((q-l)^2 - m_\pi^2),$$

which can be obtained from Fig. 2 by cutting through the propagators (using Cutkosky rules). Based on the "Form Factors + Disp Theory. pdf" notes from Bashran, we will in the following eliminate the δ -distributions in the expression on the right-hand side (RHS) and rewrite the discontinuity equation in terms of $F(s, z_s)$. Note that

$$M(s, t, u) = i \epsilon_{\mu\nu\alpha\beta} n^\mu p_+^\nu p_-^\alpha p_0^\beta F(s, t, u),$$

so that

$$i \epsilon_{\mu\nu\alpha\beta} n^\mu p_+^\nu p_-^\alpha p_0^\beta \text{disc } F(s, z_s) = \frac{i}{8\pi^2} \int d^4 l \left[i \epsilon_{\mu\nu\alpha\beta} n^\mu (q-l)^\nu l^\alpha p_0^\beta F(s, z_s') \right] T^{*} (s, z_s'') \times \delta(l^2 - m_\pi^2) \delta((q-l)^2 - m_\pi^2),$$

where $q = p_+ + p_-$, $z_s = \cos \theta_s^{(ii)}$, and $\theta_s^{(ii)}$ denote the scattering angles between the initial- and final-state momenta, between the initial and intermediate state, and between the intermediate and final state, respectively.

Why can we drop n^μ ?
 See also Schneider, where this is solved really weirdly (wrong? does not give the correct result and misses $\epsilon_{\mu\nu\alpha\beta}$ part of the term?)

also Dropping the polarization vector n^μ (see also Bashran's notes), we obtain:

$$i \epsilon_{\mu\nu\alpha\beta} p_+^\nu p_-^\alpha p_0^\beta \text{disc } \mathcal{F}(s, z_s)$$

$$= \frac{i}{8\pi^2} \int d^4\ell \left[i \epsilon_{\mu\nu\alpha\beta} (q-\ell)^\nu \ell^\alpha p_0^\beta \mathcal{F}(s, z_s^i) \right] T^{1*}(s, z_s^u) \delta(\ell^2 - m^2) \delta((q-\ell)^2 - m^2)$$

$$= i \epsilon_{\mu\nu\alpha\beta} q^\nu \ell^\alpha p_0^\beta \mathcal{F}(s, z_s^i) \text{ due to antisymmetry}$$

$$\bullet q = p_+ + p_- \rightarrow q^2 = s$$

$$\bullet \text{ In the CMS, we have } q^0 = \sqrt{s}$$

$$\bullet \delta(\ell^2 - m^2) = \delta(\ell^0)^2 - (\vec{\ell}^2 + m^2), \quad \delta((q-\ell)^2 - m^2) = \delta(q^2 + \ell^2 - 2q \cdot \ell - m^2)$$

$$\bullet \delta(f(x)) = \sum_x \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

$$\bullet d^4\ell = d\ell^0 d|\vec{\ell}| |\vec{\ell}|^2 d\Omega_{\vec{\ell}}$$

$$\bullet \ell^0 = \sqrt{|\vec{\ell}|^2 + m^2} \rightarrow \frac{d\ell^0}{d|\vec{\ell}|} = \frac{|\vec{\ell}|}{\ell^0} \rightarrow d\ell^0 \ell^0 = d|\vec{\ell}| |\vec{\ell}|$$

$$= \frac{i}{8\pi^2} \int d\ell^0 d|\vec{\ell}| d\Omega_{\vec{\ell}} |\vec{\ell}|^2 \left[i \epsilon_{\mu\nu\alpha\beta} q^\nu \ell^\alpha p_0^\beta \mathcal{F}(s, z_s^i) \right] T^{1*}(s, z_s^u)$$

$$\times \delta(\ell^0)^2 - (|\vec{\ell}|^2 + m^2) \delta(s - 2q^0 \ell^0)$$

$$= \frac{i}{8\pi^2} \int \frac{d|\vec{\ell}| d\Omega_{\vec{\ell}}}{2\ell^0} |\vec{\ell}|^2 \left[i \epsilon_{\mu\nu\alpha\beta} q^\nu \ell^\alpha p_0^\beta \mathcal{F}(s, z_s^i) \right] T^{1*}(s, z_s^u) \delta(s - 2\sqrt{s}\ell^0)$$

$$= \frac{i}{8\pi^2} \int \frac{d\ell^0 d|\vec{\ell}|}{2} |\vec{\ell}| \left[i \epsilon_{\mu\nu\alpha\beta} q^\nu \ell^\alpha p_0^\beta \mathcal{F}(s, z_s^i) \right] T^{1*}(s, z_s^u) \delta\left(2\sqrt{s}\left(\ell^0 - \frac{s}{2\sqrt{s}}\right)\right)$$

$$= \frac{1}{2\sqrt{s}} \frac{i}{4\pi^2} \int d\Omega_{\vec{\ell}} \sqrt{\frac{s}{4} - m^2} \left[i \epsilon_{\mu\nu\alpha\beta} q^\nu \ell^\alpha p_0^\beta \mathcal{F}(s, z_s^i) \right] T^{1*}(s, z_s^u)$$

$$\sigma_{\pi}(s) = \sqrt{1 - \frac{4m^2}{s}}$$

$$= \frac{15\sigma_{\pi}(s)}{64\pi^2} \int d\Omega_{\vec{\ell}} \left[i \epsilon_{\mu\nu\alpha\beta} q^\nu \ell^\alpha p_0^\beta \mathcal{F}(s, z_s^i) \right] T^{1*}(s, z_s^u)$$

In order to obtain an expression for $\text{disc } \mathcal{F}(s, z_s)$, we now contract both sides of the equation with $[i \epsilon_{\mu\nu\alpha\beta} p_+^\nu p_-^\alpha p_0^\beta]$, where certain scalar products of the momenta will be left. To evaluate these, we first have to work out some kinematics and to this end stick to the CMS of the $2 \rightarrow 2 \rightarrow 2$ process.

Unitarity Relation for $V \rightarrow 3\pi$ ($V \rightarrow \pi\pi\pi$) 2

We define $p_v = \begin{pmatrix} E_v \\ \vec{p}_v \end{pmatrix}$, $p_0 = \begin{pmatrix} E_0 \\ -\vec{p}_0 \end{pmatrix}$,

$$q-l = \begin{pmatrix} E_{q-l} \\ \vec{p}_{q-l} \end{pmatrix}, \quad l = \begin{pmatrix} E_l \\ -\vec{p}_l \end{pmatrix},$$

$$p_+ = \begin{pmatrix} E_+ \\ \vec{p}_+ \end{pmatrix}, \quad p_- = \begin{pmatrix} E_- \\ -\vec{p}_- \end{pmatrix}, \quad \text{where (see Mechanica)}$$

$$|\vec{p}_v| = \frac{\sqrt{\lambda(s, M_v^2, M_\pi^2)}}{2\sqrt{s}}, \quad E_v = \frac{M_v^2 - M_\pi^2 + s}{2\sqrt{s}},$$

$$E_0 = \frac{M_v^2 - M_\pi^2 - s}{2\sqrt{s}},$$

$$|\vec{p}_{q-l}| = \frac{\sqrt{s}}{2} \sigma_\pi(s),$$

$$E_{q-l} = \frac{\sqrt{s}}{2} = E_l,$$

$$|\vec{p}_{\pi\pi}| = \frac{\sqrt{s}}{2} \sigma_\pi(s),$$

$$E_+ = \frac{\sqrt{s}}{2} = E_-.$$

The angles were defined such that

$$\vec{p}_v \cdot \vec{p}_\pi = |\vec{p}_v| |\vec{p}_\pi| \cos \theta_s,$$

$$\vec{p}_v \cdot \vec{p}_{\pi\pi} = |\vec{p}_v| |\vec{p}_{\pi\pi}| \cos \theta_s', \quad (*)$$

$$\vec{p}_{\pi\pi} \cdot \vec{p}_\pi = |\vec{p}_{\pi\pi}| |\vec{p}_\pi| \cos \theta_s''.$$

With $\vec{p}_v = |\vec{p}_v| \begin{pmatrix} \sin \theta_1 \cos \phi_1 \\ \sin \theta_1 \sin \phi_1 \\ \cos \theta_1 \end{pmatrix}$, $\vec{p}_{\pi\pi} = |\vec{p}_{\pi\pi}| \begin{pmatrix} \sin \theta_2 \cos \phi_2 \\ \sin \theta_2 \sin \phi_2 \\ \cos \theta_2 \end{pmatrix}$,

$$\vec{p}_\pi = |\vec{p}_\pi| \begin{pmatrix} \sin \theta_3 \cos \phi_3 \\ \sin \theta_3 \sin \phi_3 \\ \cos \theta_3 \end{pmatrix},$$

We align $\theta_1 = 0$ and $\phi_3 = 0$, such that comparing with (*) yields

$$\vec{p}_v = |\vec{p}_v| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{p}_{\pi\pi} = |\vec{p}_{\pi\pi}| \begin{pmatrix} \sin \theta_s' \cos \phi_s' \\ \sin \theta_s' \sin \phi_s' \\ \cos \theta_s' \end{pmatrix},$$

$$\vec{p}_\pi = |\vec{p}_\pi| \begin{pmatrix} \sin \theta_s \\ 0 \\ \cos \theta_s \end{pmatrix},$$

With the additional constraint that

$$\frac{\vec{P}_1 \cdot \vec{P}_2}{|\vec{P}_1| |\vec{P}_2|} = \cos \theta_s'' \quad , \quad \text{i.e.} \quad \cos \theta_s'' = \sin \theta_s \sin \theta_s' \cos \phi_s' + \cos \theta_s \cos \theta_s'.$$

Performing the aforementioned contraction with Mathematica, we ultimately obtain

$$\text{disc } F(s, z_s) = \frac{i\sigma_\pi(s)}{64\pi^2} \int_{-1}^1 d\cos \theta_s' \int_0^{2\pi} d\phi_s' \frac{\cos \theta_s'' - \cos \theta_s \cos \theta_s'}{1 - \cos \theta_s} F(s, z_s') T^{1*}(s, z_s'')$$

Using the partial-wave expansions for $F(s, z_s^{(i)})$ and $T^{1*}(s, z_s^{(i)})$,

$$F(s, t, u) = \sum_{\ell \text{ odd}} f_\ell(s) P_\ell'(z_s),$$

$$T^\mp(s, t, u) = 32\pi \sum_{\ell=0}^{\infty} (2\ell+1) t_\ell^\mp(s) P_\ell(z_s),$$

$$t_\ell^\mp(s) = \frac{e^{2i\delta_\ell^\mp(s)} - 1}{2i\sigma_\pi(s)} = e^{i\delta_\ell^\mp(s)} \frac{\sin \delta_\ell^\mp(s)}{\sigma_\pi(s)},$$

$$\left(\text{from AOT: } t_\ell^\mp(s) = \frac{e^{2i\delta_\ell^\mp(s)} - 1}{2ik} \right),$$

$k = \frac{\sqrt{s}}{2} \sigma_\pi(s)$
So different definition in AOT?

We can project out the $\ell=1$ partial wave to arrive at a Watson-like unitarity relation ($\mathbb{I}=1$ due to $\mathbb{I}=1$ in initial state),

$$\begin{aligned} \text{disc } f_1(s) &= \frac{1}{2} \int_{-1}^1 dz_s [P_0(z_s) - P_2(z_s)] \text{disc } F(s, z_s) \\ &= \frac{1}{2} \int_{-1}^1 dz_s [P_0(z_s) - P_2(z_s)] \frac{i\sigma_\pi(s)}{64\pi^2} \int_{-1}^1 dz_s' \int_0^{2\pi} d\phi_s' \frac{z_s'' - z_s z_s'}{1 - z_s^2} \\ &\quad \times \left[\sum_{\ell \text{ odd}} f_\ell(s) P_\ell'(z_s') \right] \left[32\pi \sum_{j=0}^{\infty} (2j+1) t_j^{1*} P_j(z_s'') \right] \\ &= \frac{i\sigma_\pi(s)}{4\pi} \sum_{\ell \text{ odd}} \sum_{j=0}^{\infty} f_\ell(s) (2j+1) t_j^{1*}(s) \int_{-1}^1 dz_s \int_{-1}^1 dz_s' \int_0^{2\pi} d\phi_s' [P_0(z_s) - P_2(z_s)] \\ &\quad \times \frac{z_s'' - z_s z_s'}{1 - z_s^2} P_\ell'(z_s') P_j(z_s'') \\ &= \frac{\sqrt{1-z_s^2} \sqrt{1-z_s'^2} \cos \phi_s'}{1 - z_s^2} \end{aligned}$$

Can include $\delta(s - 4m^2)$ because we cut propagators \rightarrow poles on shell (but already in Eqs. before?)

Okay to check with Mathematica for $L \leq 5$

Mathematica up to $\ell=5=j$

$$= \frac{i\sigma_\pi(s)}{4\pi} \sum_{\ell=1}^{\infty} \sum_{j=0}^{\infty} f_\ell(s) (2j+1) t_j^{1*}(s) \left[\frac{8\pi}{3} \tau_{\ell j} \delta_{\ell j} \right] = 2i f_1(s) \sin^2 \theta_s' e^{-i\delta_1'(s)} \theta(s - 4m^2)$$

Unitarity Relation for $V \rightarrow 3\pi$ ($N\pi \rightarrow 3\pi$)

We now note that $\text{disc } F(s) = \text{disc } f_1(s)$, i.e. the discontinuity is entirely contained in the P -wave, ^{and we} can write (s-channel) \nearrow

just mathematically
 $f_1(s) = F(s) + \hat{F}(s)$, where $\text{disc } \hat{F}(s) = 0 \forall s \geq 4m_\pi^2$

(See also the Diplana thesis from Franz).

Here, $\hat{F}(s)$ is called the inhomogeneity.

In order to express $f_1(s)$ in terms of $F(s)$, we can use the partial-wave decomposition,

for $0(s < 4m_\pi^2)$
 $f_1(s) = \frac{1}{2} \int_{-1}^1 dz_3 [P_0(z_3) - P_2(z_3)] F(s, t, u)$

$F(s, t, u) = F(s) + F(t) + F(u)$

neglecting F - and higher partial waves

$= \frac{1}{2} \int_{-1}^1 dz_3 [P_0(z_3) - P_2(z_3)] (F(s) + F(t) + F(u))$

$P_0(z_3) = 1, P_2(z_3) = \frac{1}{2}(3z_3^2 - 1)$

$t = \frac{3s_0 - s + k(s)z_3}{2}$, (See Diplana thesis from Franz)

$u = \frac{3s_0 - s - k(s)z_3}{2}$

$3s_0 = s + t + u = M_V^2 + 3M_\pi^2, k(s) = \sigma_\pi(s) \sqrt{\lambda(M_V^2, M_\pi^2, s)}$

$= \frac{1}{2} \int_{-1}^1 dz_3 \left[\frac{3}{2} - \frac{3}{2} z_3^2 \right] (F(s) + F(t) + F(u))$

$= \frac{3}{4} \int_{-1}^1 dz_3 (1 - z_3^2) [F(s) + F(t) + F(u)]$

$= F(s) + \frac{3}{4} \int_{-1}^1 dz_3 (1 - z_3^2) F\left(\frac{3s_0 - s + k(s)z_3}{2}\right)$

$+ \frac{3}{4} \int_{-1}^1 dz_3 (1 - z_3^2) F\left(\frac{3s_0 - s - k(s)z_3}{2}\right)$

is this the proper explanation for disc f(s) = disc F(s)?
 just mathematically
 from $f_1(s) = \frac{1}{2} \int_{-1}^1 dz_3 [P_0(z_3) - P_2(z_3)] F(s, t, u)$
 and $F(s)$ only has a right-hand cut? (No cut)
 $P_0(z) = F(s) + \hat{F}(s)$ is a relation in "Extracting the direct channel" from $3\pi \rightarrow 3\pi$ paper.
 is this then only true for $0(s < 4m_\pi^2)$ because otherwise, disc f(s) also contains the LHC(s)? If so, why does unitarity relation for f(s) not include the LHC?

$$\begin{aligned}
 & z_s \mapsto -z_s \text{ in one of the last two integrals} \\
 & = F(s) + \frac{3}{2} \int_{-1}^1 dz_s (1-z_s^2) F\left(\frac{3s_0 - s + 2(s)z_s}{2}\right) \\
 & = F(s) + 3 \langle (1-z_s)^2 F \rangle (s),
 \end{aligned}$$

$$\text{where } \langle z_s^n f \rangle (s) = \frac{1}{2} \int_{-1}^1 dz_s z_s^n f\left(\frac{3s_0 - s + 2(s)z_s}{2}\right).$$

We can thus identify $\tilde{F}(s) = 3 \langle (1-z_s)^2 F \rangle (s)$, which contains the left-hand cut contribution to the partial wave $f_{\ell}(s)$.
 With this, the Watson-like unitarity relation becomes

$$\text{disc } F(s) = 2i (F(s) + \tilde{F}(s)) \theta(s - 4m^2) \approx \delta(s) e^{-i\delta(s)},$$

where $\delta(s) = \delta_{\ell}(s)$ is the π P-wave phase shift.

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In order to solve the obtained unitarity relation, we start by rewriting it according to (see Franz' Diploma thesis)

$$\frac{F(s_+) - F(s_-)}{2i} = (F(s_+) + F(s_-)) \theta(s - 4m_\pi^2) \sin \delta(s) e^{-i\delta(s)}$$
$$= (F(s_+) + F(s_-)) \theta(s - 4m_\pi^2) \frac{1 - e^{-2i\delta(s)}}{2i},$$

where we used that the physical amplitude is obtained by taking the limit on the cut from above, then

$$F(s_+) - F(s_-) = \left\{ F(s_+) (1 - e^{-2i\delta(s)}) + F(s_+) (2i \sin \delta(s) e^{-i\delta(s)}) \right\} \theta(s - 4m_\pi^2)$$

suppressing the $\theta(s - 4m_\pi^2) \times \theta(s - 4m_\pi^2)$

$$\Leftrightarrow F(s_+) e^{-2i\delta(s)} - F(s_-) = 2i F(s_+) \sin \delta(s) e^{-i\delta(s)} \quad (*)$$

This is an inhomogeneous Hilbert-type equation, whose solutions are well known.

Before considering the full solution, we take a look at the homogeneous problem, $F(s) = 0$. (Note that in this case, the unitarity relation is a manifestation of Watson's final-state theorem: the form factor shares the phase of the (elastic) scattering amplitude.) The solution to this problem is given in terms of the Omnès function (multiplied by a polynomial), which can be seen as follows:

$$(*) \Rightarrow \log(F(s_+)) - 2i\delta(s) = \log(F(s_-))$$

$$\Leftrightarrow \log(F(s_+)) - \log(F(s_-)) = 2i\delta(s) \theta(s - 4m_\pi^2).$$

↑
introduce $\theta(s - 4m_\pi^2)$ back into the equation

Thus, we have

$$\text{disc}[\log(F(s))] = 2i\delta(s)\theta(s-4M^2)$$

and hence from a once-subtracted dispersion integral relation

$$F(s) = P(s)\Omega(s)$$

$$\Omega(s) = \exp\left\{\frac{s}{\pi} \int_{4M^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\right\},$$

where $P(s)$ is a polynomial and the Omnès function is normalized to $\Omega(0) = 1$ (i.e. no subtraction constant appears).

The Omnès function furthermore fulfills

$$\Omega(s_+) = |\Omega(s)| e^{i\delta(s)}, \quad \Omega(s_-) = |\Omega(s)| e^{-i\delta(s)},$$

which can be seen via various ways:

$$1) \text{disc } F(s) = 2i \lim_{\epsilon \rightarrow 0} F(s) = 2i (F(s) + \cancel{F(s)}) \theta(s-4M^2) s \delta(s) e^{-i\delta(s)}$$

$$\lim_{\epsilon \rightarrow 0} F(s) \in \mathbb{R} \Rightarrow F(s) = |F(s)| e^{i\delta(s)}$$

evaluated above the cut, $s = s_+$, (physical region)

$$(*) \Leftrightarrow \Omega(s_+) e^{-2i\delta(s)} = F(s_-)$$

$$\Leftrightarrow \Omega(s) = |F(s)| e^{-i\delta(s)} \quad (\text{up to a multiplicative polynomial});$$

$$2) \text{ Using Schwarz' reflection principle, } \Omega(s_+) = \Omega^*(s_-),$$

$$\Omega(s_+) = |\Omega(s_+)| e^{i\delta(s)} \Rightarrow \Omega(s_-) = |\Omega(s_+)| e^{-i\delta(s)}$$

3) Using the Sokhotski-Remelj theorem,

$$\Omega(s_{\pm}) = \exp\left\{\frac{s}{\pi} \int_{4M^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s_{\pm}i\epsilon)}\right\}$$

$$\int_{4M^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s_{\pm}i\epsilon)} \stackrel{s'=s-s}{=} \int_{4M^2-s}^{\infty} d\tilde{s} \frac{\delta(\tilde{s}+s)}{(\tilde{s}+s)(\tilde{s}_{\pm}i\epsilon)} = \pm i\pi \frac{\delta(s)}{s} + \text{PV} \int_{4M^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}$$

$$= \exp\left\{\pm i\pi\right\} \exp\left\{\frac{s}{\pi} \text{PV} \int_{4M^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\right\}$$

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$$= |R(s_{\pm})| e^{\pm i\delta(s)} = |R(s)| e^{\pm i\delta(s)}$$

$$\uparrow |R(s_{\pm})| = |R^*(s_{\pm})| = |R(s_{\pm})| = |R(s)|$$

↑ Schwarz's reflection principle or 1)

$$4) \quad R(s) = \sqrt{R(s_+)^2} \stackrel{(*)}{=} \sqrt{R(s_+)R(s_-)} e^{i\delta(s)} \stackrel{\downarrow}{=} |R(s)| e^{i\delta(s)}$$

$$R(s) = R(s_+) e^{-2i\delta(s)} = |R(s)| e^{-i\delta(s)}$$

Note that the asymptotic behavior of the Omnès function is constrained by the one of the phase shift. For $\delta(s \rightarrow \infty) = k\pi$ one finds $R(s) \rightarrow s^{-k}$, namely

$$R(s) = \exp \left\{ k \int_{4m^2}^{\infty} ds' \frac{s-s'+s'}{s'(s'-s)} \right\} = \exp \left\{ k \int_{4m^2}^{\infty} ds' \left[\frac{s'}{s'(s'-s)} - \frac{s'-s}{s'(s'-s)} \right] \right\}$$

$$= \exp \left\{ k \int_{4m^2}^{\infty} ds' \left[\frac{1}{s'-s} - \frac{1}{s'} \right] \right\} = \exp \left\{ k \left[\log|s'-s| - \log|s'| \right] \Big|_{4m^2}^{\infty} \right\}$$

$$= \exp \left\{ k \log \left| 1 - \frac{s}{s'} \right| \Big|_{4m^2}^{\infty} \right\} = \exp \left\{ k \left[\log|1| - \log \left| 1 - \frac{s}{4m^2} \right| \right] \right\}$$

$$= \left| 1 - \frac{s}{4m^2} \right|^{-k} \rightarrow s^{-k}$$

(For an alternative derivation, see my Bachelor's thesis.)

In the following, $k=1$ is assumed for the asymptotic behavior of the P-wave (this guarantees the high-energy fall-off for the form factor suggested by PQCD ($\sim 1/s$), exactly if $P(s)$ is a constant; hence, due to gauge invariance $P(s)=1$).

In order to suppress the high-energy behavior of the $\pi\pi$ phase shift (which is not known to arbitrarily high energies), one can resort to a twice-subtracted dispersion integral

$$R(s) = \exp \left\{ \frac{1}{f_0} \langle r^2 \rangle \frac{V}{\pi} s + \frac{s^2}{\pi} \int_{4m^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)} \right\}$$

where the additional subtraction constant is related to the charge radius of the pion due to the identification $F_{\pi}^V(s) = \Omega(s)$.

The once- and twice-subtracted dispersion relation can be compared by noting that

$$\frac{s}{s'(s'-s)} = \frac{s}{s'^2} + \frac{s^2}{(s'-s)s'^2} \quad \left(\text{see Melnikov's Master's thesis} \right), \text{ i.e.}$$

$$\Omega(s) = \exp \left\{ \frac{s}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \frac{\rho(s')}{s'(s'-s)} \right\} = \exp \left\{ \frac{s}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \frac{\rho(s')}{s'^2} + \frac{s^2}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \frac{\rho(s')}{s'^2(s'-s)} \right\}$$

resulting in a sum rule for $\langle r^2 \rangle_{\pi}^V$, namely

$$\langle r^2 \rangle_{\pi}^V = \frac{6}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \frac{\rho(s')}{s'^2}.$$

(To take advantage of the suppression of high energies in the oversubtracted dispersion integral, one may make use of an independent (phenomenological) determination of the charge radius; note, however, that in principle, using a charge radius different from the sum-rule value is inconsistent and leads to a wrong high-energy behavior.)

To find a solution for the full unitarity relation, one derives an integral equation for $F(s)/\Omega(s)$. Indeed, we make the ansatz (see Franz' Diploma thesis) $F(s) = \Omega(s)\phi(s)$, which, when inserted into (*) yields

$$\Omega(s_+) \phi(s_+) e^{-2i\delta(s)} - \Omega(s_-) \phi(s_-) = 2i \hat{F}(s_+) s \sin\delta(s) e^{-i\delta(s)}$$

$$\Leftrightarrow [\phi(s_+) - \phi(s_-)] \Omega(s_-) = 2i \hat{F}(s_+) s \sin\delta(s) e^{-i\delta(s)}$$

$$\hookrightarrow \text{disc } \phi(s) = \frac{2i \hat{F}(s_+) s \sin\delta(s) e^{-i\delta(s)}}{-\Omega(s_-)} \theta(s - 4m_{\pi}^2)$$

↑
insert $\theta(s - 4m_{\pi}^2)$
back into eq.

$$= \frac{2i \hat{F}(s) s \sin\delta(s)}{|\Omega(s)|} \theta(s - 4m_{\pi}^2)$$

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We can thus rewrite $\phi(s)$ using a dispersion relation,

$$\phi(s) = P_n + \frac{s^n}{\pi} \int_{4m^2}^{\infty} ds' \frac{\hat{F}(s') \sin \delta(s')}{|Q(s')| \sin^2(s'-s)}$$

$$\Rightarrow F(s) = \Omega(s) \left\{ P_n + \frac{s^n}{\pi} \int_{4m^2}^{\infty} ds' \frac{\hat{F}(s') \sin \delta(s')}{|Q(s')| \sin^2(s'-s)} \right\}$$

What remains to be done is the number of subtractions that are necessary. From the Froissart bound, we know that $M(s, t, u) \stackrel{s \text{ large}}{<} C s \log^2 s$, so that with

$$|M(s, t, u)|^2 = \frac{s}{16} \underbrace{k^2(s)}_{\sim s^2} \sin^2 \theta_s |F(s, t, u)|^2,$$

it follows that $F(s, t, u) < \tilde{C} s^{-1/2} \log^2 s$ asymptotically.

Since $F(s, t, u) = F(s) + F(t) + F(u)$, it follows that

$\lim_{s \rightarrow \infty} F(s) < \hat{C} s^{-1/2} \log^2(s)$, since otherwise, $F(t) + F(u)$ would have to cancel the contributions of higher order than $s^{-1/2} \log^2 s$, which is not the case in general (see also Franz' Diploma thesis). We thus find

$$F(s) = \Omega(s) \left\{ P_n + \frac{s^n}{\pi} \int_{4m^2}^{\infty} ds' \frac{\hat{F}(s') \sin \delta(s')}{|Q(s')| \sin^2(s'-s)} \right\} < \hat{C} s^{-1/2} \log^2(s).$$

Likewise to before, one can argue that $\hat{F}(s) < C' s^{-1/2} \log^2(s)$ having the same asymptotic behavior as $F(s)$ (see Franz' Diploma thesis, where it is stated that one could also expect this intuitively; note also how $\hat{F}(s) = 3 \langle (1-2s^2) F \rangle (s)$ should exhibit the same asymptotic behavior).

where does this boundary or the integral come from? Or $1/s!$ correct for $1/s!$ $= \log(s)^2$

Noting that the integrand of the above dispersion relation for $F(s)$

needs to decrease stronger than $1/s$ for the integral to converge, we find asymptotically (for all $s \in \mathbb{R}^+$)

$$\lim_{s' \rightarrow \infty} \frac{F(s') \sin \delta(s')}{\Omega(s') |s'|^n (s'-s)} = \text{const} \lim_{s' \rightarrow \infty} \frac{F(s')}{|s'|^{n-k} (s'-s)}$$

$$< \text{const} \lim_{s' \rightarrow \infty} \frac{\log^2(s')}{|s'|^{n-k+1/2} (s'-s)} ;$$

$$\bullet k=1, n=0: \lim_{s' \rightarrow \infty} \frac{\log^2(s')}{|s'|^{1-1/2} (s'-s)} = \lim_{s' \rightarrow \infty} \frac{\sqrt{s'} \log^2(s')}{s'-s}$$

$$\stackrel{\text{L'Hospital}}{=} \lim_{s' \rightarrow \infty} \frac{\frac{1}{2\sqrt{s'}} \log^2(s') + 2\sqrt{s'} \log(s') \frac{1}{s'}}{1} = \lim_{s' \rightarrow \infty} \frac{1}{\sqrt{s'}} \left(\frac{\log^2(s')}{2} + 2\log(s') \right),$$

which does not decrease fast enough;

$$\bullet k=1, n=1: \lim_{s' \rightarrow \infty} \frac{\log^2(s')}{|s'|^{1-1/2} (s'-s)} \stackrel{\text{L'Hospital}}{=} \lim_{s' \rightarrow \infty} \frac{2\log(s') \frac{1}{s'}}{\frac{1}{2\sqrt{s'}} (s'-s) + \sqrt{s'}}$$

$$= \lim_{s' \rightarrow \infty} \frac{2\log(s')}{\frac{\sqrt{s'}}{2} (s'-s) + s'^{3/2}},$$

which is sufficient.

We can thus use the once-subtracted dispersion integral

$$F(s) = \Omega(s) \left\{ a + \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s') F(s')}{\Omega(s') |s'-s|} \right\}$$

Similar to what was done for the amplitudes function, and to suppress the influence of inelastic contributions even further, one can subtract the dispersive solution once more than strictly necessary, at the expense of introducing another subtraction constant:

$$F(s) = \Omega(s) \left\{ a + b's + \frac{s^2}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s') F(s')}{\Omega(s') |s'-s|} \right\}$$

which is equivalent to the once-subtracted version (as it should)

$$\text{if } F(s) = \Omega(s) \left\{ a + \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s') F(s')}{\Omega(s') |s'-s|} \right\} = \Omega(s) \left\{ a + \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s') F(s')}{\Omega(s') |s'-s|} \right. \\ \left. + \frac{s^2}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s') F(s')}{\Omega(s') |s'-s|} \right\}, \text{ i.e. } b' = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s') F(s')}{\Omega(s') |s'-s|}$$

Unitarity Relation for $V \rightarrow 3\pi$ ($V \rightarrow \pi\pi$) \neq

23.04.2021

In the following, we want to briefly examine the size of higher partial waves.

We start by noting that projecting the t- and u-channel P-wave amplitudes onto the F-wave yields a non-vanishing contribution (which is real, see Paper; here, we do not calculate it explicitly). Indeed, we find

$$\begin{aligned}
 f_3(s) &= \frac{1}{2} \int_{-1}^1 dz_3 \left[P_2(z_3) - P_4(z_3) \right] F(s, t, u) \\
 &= \frac{1}{2} (3z_3^2 - 1) - \frac{1}{8} (35z_3^4 - 30z_3^2 + 3) \\
 &= -\frac{35}{8} z_3^4 + \frac{21}{4} z_3^2 - \frac{7}{8} \\
 &= -\frac{7}{16} \int_{-1}^1 dz_3 \left[5z_3^4 - 6z_3^2 + 1 \right] (F(s) + F(t) + F(u)) \\
 &= -\frac{7}{16} \int_{-1}^1 dz_3 \left[5z_3^4 - 6z_3^2 + 1 \right] (F(s) + F(u)),
 \end{aligned}$$

i.e. the s-channel contribution itself (of course) vanishes.

The decomposition $F(s, t, u) = F(s) + F(t) + F(u)$ is now amended according to

$$F(s, t, u) = F(s) + F(t) + F(u) + P_3'(z_s) G(s) + P_3'(z_t) G(t) + P_3'(z_u) G(u)$$

Where is this decomposition coming from? Why include the $P_3'(z)$ factors? Note that $P_3'(z) = \dots$ so that this factor also exists for $F(\dots)$. Or just from $F(s, t, u) = F(s)P_3'(z_s) + P_3'(z_t)G(t) + P_3'(z_u)G(u)$ and then sort of subtracting over all channels?

in order to also include F-wave discontinuities; here, $G(s)$ again only has a right-hand cut and

$$z_t = \frac{s-u}{2t}, \quad z_u = \frac{s-t}{2u}.$$

As before, the discontinuity of the P-wave is expressed by

$$\text{disc } f_3(s) = \text{disc } F(s) = 2s (F(s) + F(s)) \theta(s - 4m_\pi^2) \text{ sinc}_1^1(s) e^{-i\text{d}_1^1(s)}.$$

For the F-wave, we obtain a similar expression by

projecting the $L=3$ wave out of the initial unitarity relation:

$$\text{disc } f_3(s) = \frac{i\sigma_3(s)}{4\pi} \sum_{\ell \text{ odd}} \sum_{j=0}^{\ell} f_{\ell}(s) (2j+1) t_j^{*}(s) \left[\frac{8\pi}{7} \delta_{\ell 3} \delta_{j3} \right]$$

$$= 2i f_3(s) s \sin^2 \delta_3^1(s) e^{-i\delta_3^1(s)} \theta(s - 4M^2),$$

where $\delta_3^1(s)$ is the $\pi\pi$ F-wave phase shift. Using $\text{disc } f_3(s) = \text{disc } G(s)$ and $f_3(s) = G(s) + \hat{G}(s)$, where $\text{disc } \hat{G}(s) = 0 \forall s \geq 4M^2$, we thus have

$$\text{disc } f_3(s) = \text{disc } G(s) = 2i (G(s) + \hat{G}(s)) \theta(s - 4M^2) s \sin^2 \delta_3^1(s) e^{-i\delta_3^1(s)}.$$

In order to determine the inhomogeneities, we proceed as before:

$$f_1(s) = \frac{3}{4} \int_{-1}^1 dz_3 (1-z_3^2) \left[F(s) + F(t) + F(u) + P_3'(z_3) G(s) + P_3'(z_t) G(t) + P_3'(z_u) G(u) \right]$$

$$= F(s) + \frac{3}{4} \int_{-1}^1 dz_3 (1-z_3^2) \left[F\left(\frac{3s_0 - s + k(s)z_3}{2}\right) + F\left(\frac{3s_0 - s - k(s)z_3}{2}\right) \right]$$

$$+ \frac{3}{4} \int_{-1}^1 dz_3 (1-z_3^2) \left[P_3'\left(\frac{s-t}{k(t)}\right) G\left(\frac{3s_0 - s + k(s)z_3}{2}\right) + P_3'\left(\frac{s-t}{k(u)}\right) G\left(\frac{3s_0 - s - k(s)z_3}{2}\right) \right]$$

Can express t and u in terms of s and z_3 again (*)

Can substitute $z_3 \rightarrow -z_3$ in "some" of these integrals, which also results in $t \leftrightarrow u$ and thus

$$= F(s) + \frac{3}{2} \int_{-1}^1 dz_3 (1-z_3^2) F\left(\frac{3s_0 - s + k(s)z_3}{2}\right)$$

$$+ \frac{3}{2} \int_{-1}^1 dz_3 (1-z_3^2) P_3'\left(\frac{s-t}{k(t)}\right) G\left(\frac{3s_0 - s + k(s)z_3}{2}\right)$$

(*)

$$= F(s)$$

$$+ \underbrace{3 \langle (1-z_3^2) (F + P_3' G) \rangle (s)}_{= F(s)},$$

$$= F(s)$$

where the underlining now indicates to take $z_t(s, z_3)$ as the argument of P_3' .

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$$\begin{aligned}
 f_3(s) &= -\frac{7}{16} \int_{-1}^1 dz_3 [5z_3^4 - 6z_3^2 + 1] \left[\mathcal{F}(s) + \mathcal{F}(t) + \mathcal{F}(u) \right. \\
 &\quad \left. + P_3'(z_3) G(s) + P_3'(z_3) G(t) + P_3'(z_3) G(u) \right] \\
 &= -\frac{7}{16} \int_{-1}^1 dz_3 [5z_3^4 - 6z_3^2 + 1] \left[\mathcal{F}\left(\frac{3s_0 - s + k(s)z_3}{2}\right) + \mathcal{F}\left(\frac{3s_0 - s - k(s)z_3}{2}\right) \right. \\
 &\quad \left. + G(s) - \frac{7}{16} \int_{-1}^1 dz_3 [5z_3^4 - 6z_3^2 + 1] \left[P_3'\left(\frac{s-u}{k(t)}\right) G\left(\frac{3s_0 - s + k(s)z_3}{2}\right) \right. \right. \\
 &\quad \left. \left. + P_3'\left(\frac{s-t}{k(u)}\right) G\left(\frac{3s_0 - s - k(s)z_3}{2}\right) \right] \right] \\
 &\quad \text{As before, } z_3 \mapsto -z_3 \text{ in some integrals (} t \leftrightarrow u \text{)} \\
 &= -\frac{7}{8} \int_{-1}^1 dz_3 [5z_3^4 - 6z_3^2 + 1] \mathcal{F}\left(\frac{3s - s + k(t)z_3}{2}\right) \\
 &\quad + G(s) - \frac{7}{8} \int_{-1}^1 dz_3 [5z_3^4 - 6z_3^2 + 1] P_3'\left(\frac{s-u}{k(t)}\right) G\left(\frac{3s_0 - s + k(s)z_3}{2}\right) \\
 &= G(s) - \frac{7}{4} \langle (5z_3^4 - 6z_3^2 + 1) (\mathcal{F} + P_3' G) \rangle (s) \\
 &= \hat{G}(s)
 \end{aligned}$$

26.04.2021

In order to estimate the impact of the S_3 , a Lagrangian model is used. The S_3 is described in terms of a totally symmetric third-rank tensor field $S_{\mu\nu\lambda} = S_{\mu\nu\lambda}^a \tau^a$, subject to the constraints $\partial^\mu S_{\mu\nu\lambda} = 0$, $g^{\mu\nu} S_{\mu\nu\lambda} = 0$. Pauli matrices

The simplest S_3 interaction Lagrangian will not and now takes the form

Where from is this Lagrangian the simplest exactly in this form will commutators?

$$\begin{aligned}
 \mathcal{L}_{S_3} &= \frac{g_{S_3\pi\pi}}{4F_\pi^2} \langle S_{\mu\nu\lambda} [\partial^\mu \pi, \partial^\nu \partial^\lambda \pi] \rangle \\
 &\quad + \frac{g_{S_3\pi\omega}}{2F_\pi} \epsilon^{\lambda\alpha\beta\gamma} \langle S_{\mu\nu\lambda} \partial^\mu \partial^\alpha \pi \rangle \partial^\nu \partial^\beta \omega^\gamma,
 \end{aligned}$$

where $\pi = \pi^a \tau^a$ denotes the isospin triplet of pion fields and ω the isoscalar ω vector field. Using Mathematica, we find that in write-out form, the Lagrangian becomes

$$\mathcal{L}_{S_3} = \frac{g_{S_3\pi\pi}}{F_\pi^2} i \left[S_{\mu\nu}^0 (\partial^\mu \pi^+) (\partial^\nu \pi^-) - S_{\mu\nu}^0 (\partial^\mu \pi^-) (\partial^\nu \pi^+) \right] + \frac{g_{S_3\pi\omega}}{F_\pi} e^{\lambda \alpha \beta \gamma} S_{\mu\nu}^0 (\partial^\mu \pi^+) (\partial^\nu \pi^-) (\partial^\alpha \pi^+) (\partial^\beta \pi^-) (\partial^\gamma \omega).$$

The invariant amplitude for $\pi\omega \rightarrow \pi S_3 \rightarrow 3\pi$ with this Lagrangian then reads

$$\mathcal{M}(\pi\omega \rightarrow \pi S_3 \rightarrow 3\pi) = \frac{g_{S_3\pi\omega}}{F_\pi} e^{\lambda \alpha \beta \gamma} p_0^\mu p_a^\nu p_\nu^\nu p_\nu^\nu p_\nu^\nu \eta_\gamma \frac{i p^\mu p^\nu}{\underbrace{(p_+ + p_-)^2 - M_{S_3}^2}_=s} \times \frac{i g_{S_3\pi\pi}}{F_\pi^2} (p_+^\mu p_-^\nu p_-^\alpha - p_-^\mu p_+^\nu p_+^\alpha)$$

$$\cong i \epsilon_{\mu\nu\alpha\beta} p_+^\mu p_-^\nu p_-^\alpha p_+^\beta \mathcal{F}(s, t, u).$$

Using Mathematica and the polarization sum for spin-3 particles given in the paper (see also the other notes and Mathematica file), we find an expression for $\mathcal{F}(s, t, u)$ and project out the F-wave in the s-channel (dropping an overall phase factor),

OK to drop overall phase factor (only the result from the paper?)

$$f_3(s) = \frac{g_{S_3\pi\omega} g_{S_3\pi\pi} k^2(s)}{60 F_\pi^3 (M_{S_3}^2 - s)} = C_F \frac{M_{S_3}^2}{M_{S_3}^2 - s} \frac{k^2(s)}{M_\omega^4}, \quad C_F = \frac{g_{S_3\pi\omega} g_{S_3\pi\pi} M_\omega^2}{60 F_\pi^3 M_{S_3}^2}$$

A similar expression is obtained for the P-wave and since

$$\mathcal{F}(s, t, u) = P_1'(z_5) f_1(s) + P_3'(z_5) f_3(s),$$

as readily checked with Mathematica, we indeed find that

$\mathcal{G}_{S_3}(s) = f_3(s)$, i.e. the full scalar amplitude is given by the s-channel part(s) only.

With the arguments from the paper, there is strong indication that

the $\omega \rightarrow 3\pi$ F-wave $f_3(s) = \alpha(s) + \hat{\sigma}(s)$ is dominated by the term in $\hat{\sigma}(s)$ given by the projection of the ...

Unitarity relation for $V \rightarrow 3\pi$ ($V \rightarrow \pi\pi$) g

... Crossed-channel P-wave terms. Using the scalar amplitude obtained with the hidden-local-symmetry formalism,

$$J_{HS}(s) = C_P \frac{M_\rho^2}{M_\rho^2 - s}, \quad C_P = \frac{N_c g}{8\pi^2 f_\pi^3},$$

$$N_c = 3, \quad g \approx 5.8 \text{ (universal vector coupling)},$$

We had (Mathematics)

$$\begin{aligned} \vec{G}(s) &= -\frac{7}{4} \langle (\sqrt{s} z_c^4 - 6z_c^2 + 1) (F + P_3^1 G) \rangle (s) \\ &= C_P \frac{7M_\rho^2 (-26\bar{K}^3 - 3(\bar{K}^4 - 6\bar{K}^2 + 5) \log\left(\frac{1+\bar{K}}{1-\bar{K}}\right) + 30\bar{K}}{12\bar{K}^5 (2M_\rho^2 + s - 3s_0)} \end{aligned}$$

$$= -C_P \frac{2M_\rho^2}{M_\rho^2 - \frac{1}{2}(3s_0 - s)} \frac{7}{8\bar{K}^4} \left(\frac{5 - 6\bar{K}^2 + \bar{K}^4}{2\bar{K}} \log \frac{1+\bar{K}}{1-\bar{K}} - \frac{15 - 13\bar{K}^2}{3} \right),$$

$$\text{where } \bar{K}(s) = \frac{\chi(s)}{2M_\rho^2 - 3s_0 + s}.$$

Finally, we want to derive the unitarity relation with the inelasticity parameter and its solution.

Before, we had

$$t_1^{I^*}(s) = \frac{-e^{-2i\delta(s)} + 1}{2i} = e^{-i\delta(s)} \sin\delta(s),$$

which now becomes

$$t_1^{I^*}(s) = \frac{-\eta(s)e^{2i\delta(s)} + 1}{2i},$$

$$\text{i.e. } \sin\delta(s)e^{-i\delta(s)} \rightarrow \frac{1}{2i} (1 - \eta(s)e^{-2i\delta(s)}).$$

The unitarity relation from before,

$$i(s - s_0) F(s) = 2i (F(s) + \vec{F}(s)) \rho(s - 4M_\pi^2) \sin\delta(s) e^{-i\delta(s)}$$

Why no factor $\chi(s)$ as for elastic partial wave? (where did it stem from?)

Thus becomes

$$\text{disc } \mathcal{F}(s) = (\mathcal{F}(s) + \mathcal{F}(s)) \theta(s - 4\mu_{\text{th}}^2) (1 - \eta(s) e^{-2i\delta(s)})$$

Then, $\mathcal{F}(s_+) - \mathcal{F}(s_-) = (\mathcal{F}(s_+) + \mathcal{F}(s_-)) \theta(s - 4\mu_{\text{th}}^2) (1 - \eta(s) e^{-2i\delta(s)})$

omit $\theta(s - 4\mu_{\text{th}}^2)$

$$\Leftrightarrow \mathcal{F}(s_+) \eta(s) e^{-2i\delta(s)} - \mathcal{F}(s_-) = \mathcal{F}(s_+) (1 - \eta(s) e^{-2i\delta(s)}) \quad (*)$$

For the homogeneous case, we thus have

$$\mathcal{F}(s_+) \eta(s) e^{-2i\delta(s)} = \mathcal{F}(s_-)$$

$$\Leftrightarrow \log \mathcal{F}(s_+) - \log \mathcal{F}(s_-) = 2i\delta(s) - \log \eta(s)$$

$$\Leftrightarrow \text{disc } \log \mathcal{F}(s) = 2i\delta(s) - \log \eta(s),$$

where the cut for the inelasticity $\eta(s)$ starts at $16\mu_{\text{th}}^2$. The

Solution of this equation is thus given by a "modified Omnès solution",

$$\log \mathcal{F}(s) = \frac{s}{\pi} \int_{4\mu_{\text{th}}^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)} - \frac{s}{2\pi i} \int_{16\mu_{\text{th}}^2}^{\infty} ds' \frac{\log \eta(s')}{s'(s'-s)}$$

Simply given by this expression if we have the two cuts from above?

$$\Rightarrow \mathcal{F}(s) = \Omega(s) \exp \left\{ \frac{is}{2\pi} \int_{16\mu_{\text{th}}^2}^{\infty} ds' \frac{\log \eta(s')}{s'(s'-s)} \right\} \equiv \Omega(s) \mathcal{I}(s)$$

To rewrite this with the SP formula, we have to differentiate between

the three cases i) above the cut, ii) below the cut, iii) else:

$$\mathcal{I}(s_{\pm}) = \exp \left\{ \frac{is}{2\pi} \int_{16\mu_{\text{th}}^2}^{\infty} ds' \frac{\log \eta(s')}{s'(s'-s \mp i\epsilon)} \right\} \stackrel{\tilde{s}=s-s}{=} \exp \left\{ \frac{is}{2\pi} \int_{16\mu_{\text{th}}^2-s}^{\infty} ds' \frac{\log \eta(\tilde{s}+s)}{(\tilde{s}+s)(\tilde{s} \mp i\epsilon)} \right\}$$

$$= \exp \left\{ \frac{is}{2\pi} \left[\pm i\pi \frac{\log \eta(s)}{s} + \int_{16\mu_{\text{th}}^2}^{\infty} ds' \frac{\log \eta(s')}{s'(s'-s)} \right] \right\}$$

$$= \exp \left\{ \mp \frac{1}{2} \log \eta(s) \right\} \exp \left\{ \frac{is}{2\pi} \int_{16\mu_{\text{th}}^2}^{\infty} ds' \frac{\log \eta(s')}{s'(s'-s)} \right\}$$

$$= \eta(s)^{\mp 1/2} \mathcal{I}(s)$$

$$\Rightarrow \Omega(s) = \Omega(s) \eta(s)^{\mp 1/2} \mathcal{I}(s), \quad \eta(s) = \begin{cases} \eta^{-1/2}(s), & \text{above the cut,} \\ \eta^{1/2}(s), & \text{below the cut,} \end{cases}$$

Unitarity relation for $V \rightarrow 3\pi$ ($V \rightarrow \pi\pi$) to

For the full solution of the unitarity relation (*), use (again) use a product ansatz

$$F(s) = \tilde{Q}(s) Z(s),$$

so that (*) becomes

$$\tilde{Q}(s_+) Z(s_+) \eta(s) e^{-2i\delta(s)} - \tilde{Q}(s_-) Z(s_-) = \hat{F}(s_+) (1 - \eta(s) e^{-2i\delta(s)})$$

$$\left. \begin{aligned} \tilde{Q}(s_+) &= \rho(s_+) \Xi(s) \Omega(s_+) \\ &= \eta(s)^{-1} \rho(s_-) \Xi(s) \Omega(s_-) e^{2i\delta(s)} \\ &= \eta(s)^{-1} e^{2i\delta(s)} \tilde{Q}(s_-) \end{aligned} \right\}$$

$$\Rightarrow [Z(s_+) - Z(s_-)] \tilde{Q}(s_-) = \hat{F}(s_+) (1 - \eta(s) e^{-2i\delta(s)})$$

$$\begin{aligned} \text{disc } Z(s) &= \frac{\hat{F}(s_+) (1 - \eta(s) e^{-2i\delta(s)})}{\tilde{Q}(s_-)} \\ &= \frac{\hat{F}(s_+) (1 - \eta(s) e^{-2i\delta(s)})}{\Omega(s_-) \rho(s_-) \Xi(s)} \\ &= \frac{F(s_+) (e^{i\delta(s)} - \eta(s) e^{-i\delta(s)})}{|\Omega(s)| \sqrt{\eta(s)} \Xi(s)} \end{aligned}$$

Rewriting $Z(s)$ into a dispersion relation then leads to (once-subtracted)

$$Z(s) = a + \frac{s}{2\pi i} \int_{\frac{4m^2}{s}}^{\infty} \frac{ds'}{s'} \frac{[e^{i\delta(s')} - \eta(s') e^{-i\delta(s')}] \hat{F}(s')}{\sqrt{\eta(s')} \Xi(s') |\Omega(s')| (s' - s)}$$

Wrong in paper because "a" factored out (the variable) not used $F(s)$?

$$F(s) = \rho(s) \Xi(s) \Omega(s)$$

$$\times \left\{ a + \frac{s}{2\pi i} \int_{\frac{4m^2}{s}}^{\infty} \frac{ds'}{s'} \frac{[e^{i\delta(s')} - \eta(s') e^{-i\delta(s')}] \hat{F}(s')}{\sqrt{\eta(s')} \Xi(s') |\Omega(s')| (s' - s)} \right\}$$