

Disclaimer

The notes at hand were written during my research period as a PhD student at the University of Bonn. They contain auxiliary calculations to and comments on publications by other authors, which are subject to definite conditions of use; see also the respective article(s) on <https://arxiv.org/> linked on the following website. For more information and all my material, check: <https://www.physics-and-stuff.com/>

I raise no claim to correctness and completeness of the given material!

Note that the license of the respective article(s) on <https://arxiv.org/> applies. Do not cite these notes but the original authors and the published article as given on <https://www.physics-and-stuff.com/>.

Unitarity Relation for $V \rightarrow \pi^0 e^+ e^-$

27.04.2021

In the following, we want to derive the unitarity relation given in <1206.3098> (Schneider et al.), i.e. Eq.(10) therein.

The ingredients we need are the $V(p) \rightarrow \pi^0(p_0) e^+(p_+) e^-(p_-)$ amplitude,

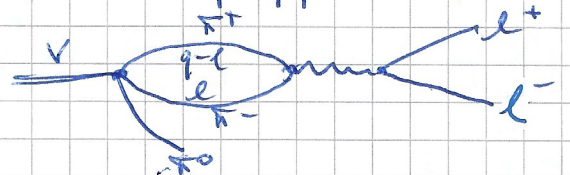
$$M_{V \rightarrow \pi^0} = ie^2 \epsilon_{\mu\nu\alpha\beta} n^\mu p_+^\nu p_-^\alpha q^\beta \frac{f_{\pi^0}(s)}{s} \bar{u}_s(p_-) \gamma^\beta v_r(p_+),$$

where $q = p_+ + p_-$, $s = (p_+ - p_-)^2 = (p_+ + p_-)^2 = q^2$, n^μ is the polarization vector of the vector meson, and $f_{\pi^0}(s)$ is the electromagnetic transition form factor of the vector meson; the $V(p) \rightarrow \pi^+(p_+) \pi^-(p_-) \pi^0(p_0)$ amplitude,

$$M_{3\pi}(s, t, u) = i \epsilon_{\mu\nu\alpha\beta} n^\mu p_+^\nu p_-^\alpha p_0^\beta F(s, t, u),$$

where $F(s, t, u)$ is the (scalar) form factor; and the pion vector form factor $F_\pi^V(s)$, defined via

$$\langle \pi^+(p) \pi^-(q) | J_\mu(s) | 0 \rangle = i (p - q)_\mu F_\pi^V(s), \quad s = (p + q)^2$$

The amplitude for  is then given by

$$iM_{V \rightarrow \pi^0} = \int \frac{d^4l}{(2\pi)^4} i \epsilon_{\mu\nu\alpha\beta} \left(\frac{1}{s} \right) \frac{i}{(q-l)^2 - m_\pi^2} \frac{i}{l^2 - m_\pi^2} (-i)(q-2l)_\mu F_\pi^V(s) \frac{-iq^{+\nu}}{q^2} e$$

$$\times (-ie) \bar{u}_s(p_-) \gamma^\nu v_r(p_+)$$

$$= e^2 \int \frac{d^4l}{(2\pi)^4} M_{3\pi}(s, t, u) \frac{1}{(q-l)^2 - m_\pi^2} \frac{1}{l^2 - m_\pi^2} (q-2l)_\mu F_\pi^V(s) \frac{1}{q^2}$$

$$\times \bar{u}_s(p_-) \gamma^\mu v_r(p_+)$$

replace prop. $\frac{1}{q^2 - m_\pi^2}$ by $(2\pi) \delta(q^2 - m_\pi^2)$

$$\text{disc } M_{V \rightarrow \pi^0} = ie^2 \int \frac{d^4l}{(2\pi)^4} M_{3\pi}(s, t, u) (2\pi) \delta((q-l)^2 - m_\pi^2) (2\pi) \delta(l^2 - m_\pi^2)$$

$$\times (q-2l)_\mu F_\pi^V(s) \frac{1}{q^2} \bar{u}_s(p_-) \gamma^\mu v_r(p_+)$$

$$= \frac{1}{s}$$

where to put factors of i? or just adjust so that the result has exactly the factor of i at the end?

Otherwise, $M_{3\pi}$ also comes in with a factor of i, but still wrong sign? or pion vector factor with F it due to c.c.? If so

Note that in F_{π^0} ! Diss. there is no i in this expression

disc. gives the wrong result on the other hand??

Inserting the amplitude on the left-hand side, we arrive at

$$iE^2 \epsilon_{\mu\nu\alpha\beta} n^\mu p_0^\nu q^\alpha \frac{1}{s} \bar{u}_s(p_-) \gamma^\beta v_r(p_+) \text{disc } f_{\text{tree}}(s)$$

$$= \frac{ie^2}{4m^2} \int d^4l M_{2\pi}(s, z_s) \delta(l^2 - M_\pi^2) \delta((q-l)^2 - m_\pi^2)$$

$$\times (q-2l)_\beta F_\pi^{V*}(s) \frac{1}{s} \bar{u}_s(p_-) \gamma^\beta v_r(p_+)$$

No add. Sing. part / disc due to the $1/s$ part in amplitude

Dividing out common "factors" and inserting the amplitude on the RHS, we find

Why here don't we divide out the spinor-part but if all common factors $E_{\text{max}} n^\mu p_0^\nu q^\alpha$ divided out, wrong result?

$$E_{\text{max}} n^\mu p_0^\nu q^\alpha \text{disc } f_{\text{tree}}(s)$$

$$= \frac{i}{4m^2} \int d^4l \left[E_{\text{max}} n^\mu (q-l)^\nu l^\alpha p_0^\beta F(s, z_s) \right] \delta(l^2 - m_\pi^2) \delta((q-l)^2 - m_\pi^2)$$

$$\times (q-2l)_\beta F_\pi^{V*}(s)$$

Suppressing the n^μ from both sides, we arrive at

$$E_{\text{max}} p_0^\nu q^\alpha \text{disc } f_{\text{tree}}(s)$$

$$= \frac{i}{4m^2} \int d^4l \left[E_{\text{max}} q^\nu l^\alpha p_0^\beta F(s, z_s) (q-2l)_\beta F_\pi^{V*}(s) \right] \delta(l^2 - m_\pi^2) \delta((q-l)^2 - m_\pi^2)$$

- $q = p_+ + p_- \implies q^2 = s$

- $q^0 = \sqrt{s}$ in the CMS

- $\delta(l^2 - m_\pi^2) = \delta((l^0)^2 - (\vec{l}^2 + m_\pi^2)) = \frac{1}{2l^0} \left[\delta(l^0 - \sqrt{\vec{l}^2 + m_\pi^2}) + \delta(l^0 + \sqrt{\vec{l}^2 + m_\pi^2}) \right]$

- $\delta((q-l)^2 - m_\pi^2) = \delta(q^2 + l^2 - 2q \cdot l - m_\pi^2) = \delta(s - 2q^0 l^0) = \delta(s - 2\sqrt{s} l^0)$
 $= \delta(2\sqrt{s} (l^0 - \frac{s}{2\sqrt{s}})) = \frac{1}{2\sqrt{s}} \delta(l^0 - \frac{\sqrt{s}}{2})$

- $d^4l = dl^0 d\vec{l} |\vec{l}|^2 d\Omega_{\vec{l}}$

- $dl^0 l^0 = d|\vec{l}| |\vec{l}|$ (from $l^0 = \sqrt{\vec{l}^2 + m_\pi^2}$)

$$= \frac{i}{4m^2} \int dl^0 d\vec{l} |\vec{l}|^2 \left[E_{\text{max}} q^\nu l^\alpha p_0^\beta F(s, z_s) (q-2l)_\beta F_\pi^{V*}(s) \right] \frac{\delta(l^0 - \sqrt{\vec{l}^2 + m_\pi^2})}{2l^0} \frac{\delta(l^0 - \frac{\sqrt{s}}{2})}{2\sqrt{s}}$$

$$= \frac{i}{4m^2} \int \frac{d|\vec{l}| |\vec{l}|}{2l^0} d\vec{l} |\vec{l}| \left[E_{\text{max}} q^\nu l^\alpha p_0^\beta F(s, z_s) (q-2l)_\beta F_\pi^{V*}(s) \right] \frac{\delta(l^0 - \frac{\sqrt{s}}{2})}{2\sqrt{s}}$$

$$= \frac{i}{8\pi^2} \int dl^0 d\Omega_{\vec{l}} |\vec{l}| \left[E_{\text{max}} q^\nu l^\alpha p_0^\beta F(s, z_s) (q-2l)_\beta F_\pi^{V*}(s) \right] \frac{\delta(l^0 - \frac{\sqrt{s}}{2})}{2\sqrt{s}}$$

Unitarity Relation for $V \rightarrow \pi^0 e^+ e^-$ 2

$$O_{\pi}(s) = \sqrt{1 - \frac{4M_{\pi}^2}{s}} \frac{i}{16\pi^2 \sqrt{s}} \int d\Omega_{\vec{z}} \sqrt{\frac{s}{4} - M_{\pi}^2} \left[E_{\mu\nu\alpha\beta} q^{\nu} z^{\alpha} p_0^{\beta} F(s, z^{\alpha}) (q-2l)_{\beta} F_{\pi}^{V*}(s) \right]$$

$$\downarrow$$

$$= \frac{i O_{\pi}(s)}{32\pi^2} \int d\Omega_{\vec{z}} \left[E_{\mu\nu\alpha\beta} q^{\nu} z^{\alpha} p_0^{\beta} F(s, z^{\alpha}) (q-2l)_{\beta} F_{\pi}^{V*}(s) \right]$$

In order to obtain an expression for disc $f_{\pi}(s)$, we now contract both sides of the equation with $[E_{\mu\nu\alpha\beta} p_0^{\nu} q^{\alpha}]$, where certain scalar products of the momenta will be left. To evaluate these, we have to work out some kinematics (see Mathematica) and ultimately arrive at

$$\text{disc } f_{\pi}(s) = \frac{i O_{\pi}(s)}{32\pi^2} \int d\Omega_{\vec{z}} \left[\frac{s - 4M_{\pi}^2}{4} (1 - \cos^2 \theta'_s) F(s, z^{\alpha}) F_{\pi}^{V*}(s) \right]$$

$$= \frac{i O_{\pi}^3(s) s}{128\pi^2} F_{\pi}^{V*}(s) \int_{-1}^1 d\cos \theta'_s \int_0^{2\pi} d\phi'_s (1 - \cos^2 \theta'_s) F(s, z^{\alpha})$$

$$= \frac{i O_{\pi}^3(s) s}{64\pi} F_{\pi}^{V*}(s) \underbrace{\int_{-1}^1 d\cos \theta'_s (1 - \cos^2 \theta'_s) F(s, z^{\alpha})}_{= \frac{4}{3} f_1(s)}$$

$$= \frac{i O_{\pi}^3(s) s}{48\pi} F_{\pi}^{V*}(s) f_1(s)$$

$$q_{\pi\pi}(s) = \frac{\sqrt{\lambda(M_{\pi}^2, M_{\pi}^2, s)}}{2\sqrt{s}} = \frac{\sqrt{s}}{2} O_{\pi}(s)$$

$$= \frac{i q_{\pi\pi}^3(s)}{6\pi \sqrt{s}} F_{\pi}^{V*}(s) f_1(s) \theta(s - 4M_{\pi}^2)$$

↑ insert to implement, where the discontinuity starts (ignored/left out before)